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Lecture 22: Pólya's Enumeration Theorem

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In this note, we introduce a powerful enumeration technique generally referred to as Pólya's enumeration theorem. Pólya's approach to counting allows us to use symmetries (such as those of geometric objects like polygons) to form generating functions. These generating functions can then be used to answer combinatorial questions such as

How many different necklaces of six beads can be formed using red, blue and green beads? What about 500-bead necklaces?

How many non-isomorphic graphs are there on four vertices? How many of them have three edges? What about on 1000 vertices with 257,000 edges? How many r-regular graphs are there on 40 vertices? (A graph is r-regular if every vertex has degree r.)

To use Pólya's techniques, we will require the idea of a permutation group. We begin with a simplified version of the first question above.

1 Coloring the Vertices of a Square

Let's begin by coloring the vertices of a square using white and red. If we fix the position of the square in the plane, there are $2^4 = 16$ different colorings. These colorings are shown in the following figure, Fig.1.

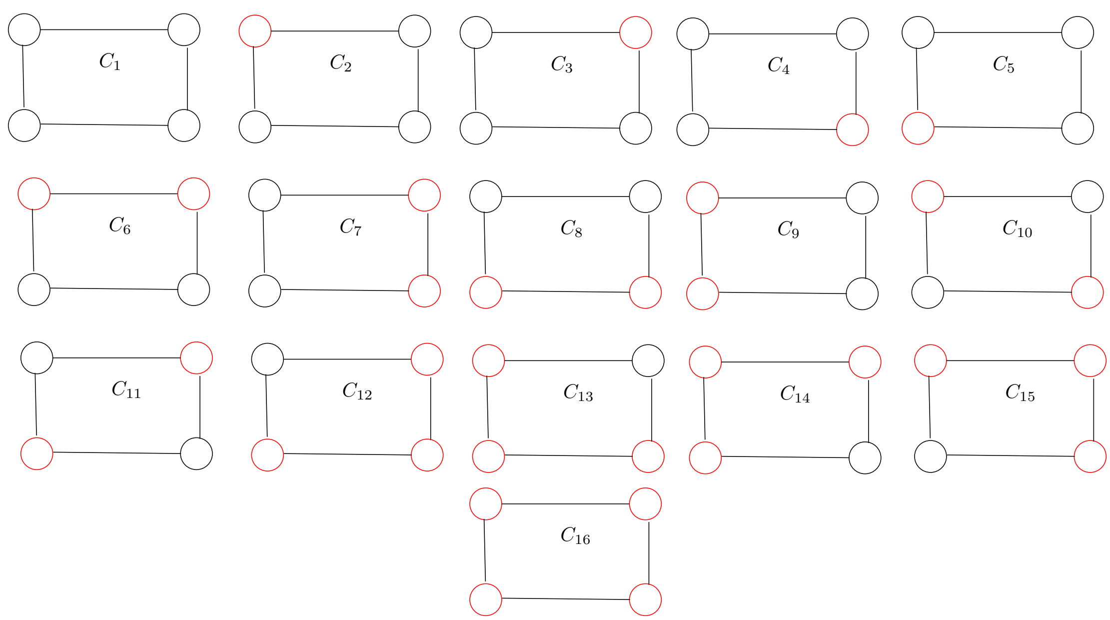

Figure 1: The 16 colorings of the vertices of a square

However, if we allow the frame to be rotated and flipped over, we realize that many of these colorings are equivalent. To systematically determine how many of the colorings shown in Fig.1 are not equivalent, we must think about the transformations we can apply to the square and what each does to the colorings. Let us make a table showing which colorings from Fig.1 are left unchanged by the application of each transformation.

Transformation	colorings unchanged
Identical transformation	All 16
Single clockwise rotation	C_1, C_{16}
Double clockwise rotation	C_1, C_{10}, C_{11}, C_{16}
Triple clockwise rotation	C_1, C_{16}
Flip on vertical line	C_1, C_6, C_8, C_{16}
Flip on horizontal line	C_1, C_7, C_9, C_{16}
Flip on positive-slope diagonal	C_1, C_3, C_5, C_{10}, C_{11}, C_{13}, C_{15}, C_{16}
Flip on negative-slope diagonal	C_1, C_2, C_4, C_{10}, C_{11}, C_{12}, C_{14}, C_{16}

It turns out that there is a useful connection between counting the nonequivalent colorings and determining the number of colorings fixed by each transformation. To develop this connection, we first need to discuss the equivalence relation created by the action of the transformations of the square on the set $\mathcal{C}$ of all 2-colorings of the square. To do this, notice that applying a transformation to a square with colored vertices results in another square with colored vertices. For instance, applying the transformation single clockwise rotation to a square colored as in C_{12} results in a square colored as in C_{13}.

Before proceeding to establish the connection between the number of nonequivalent colorings (equivalence classes under $\sim$) and the number of colorings fixed by a transformation in full generality, let's see how it looks for our example. In looking at Fig.1, you should notice that $\sim$ partitions $\mathcal{C}$ into six equivalence classes. Two contain one coloring each (the all white and all red colorings). One contains two colorings (C_{10} and C_{11}). Finally, three contain four colorings each (one red vertex, one white vertex, and the remaining four with two vertices of each color). Now look again at Table described above and add up the number of colorings fixed by each transformation. In doing this, we obtain 48, and when 48 is divided by the number of transformations (8), we get 6 (the number of equivalence classes)! It turns out that this is far from a fluke, as we will soon see. First, however, we introduce the concept of a permutation group to generalize our set of transformations of the square.

2 Permutation Groups

First, recall that a bection from a set X to itself is called a permutation.

Definition 2.1. A permutation group is a set P of permutations of a set X so that

1. the identity permutation i is in P;
2. if $\pi_1, \pi_2 \in P$, then $\pi_2 \circ \pi_1 \in P$; and
3. if $\pi_1 \in P$, then $\pi_1^{-1} \in P$.

For our purposes, X will always be finite and we will usually take $X = [n]$ for some positive integer n. The symmetric group on n elements, denoted S_n, is the set of all permutations of $[n]$. Every finite permutation group (and more generally every finite group) is a subgroup of S_n for some positive integer n.

2.1 Representing permutations

The way a permutation rearranges the elements of X is central to Pólya's enumeration theorem. A proper choice of representation for a permutation is very important here, so let's discuss how permutations can be represented. One way to represent a permutation π of $[n]$ is as a $2 \times n$ matrix in which the first row represents the domain and the second row represents π by putting $\pi(i)$ in position i. For example,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

is the permutation of $[5]$ with $\pi(1) = 2, \pi(2) = 4, \pi(3) = 3, \pi(4) = 5$, and $\pi(5) = 1$. This notation is rather awkward and provides only the most basic information about the permutation. A more compact (and more useful for our purposes) notation is known as cycle notation. We may write $\pi = (1245)(3)$ as 1 maps to 2, 2 maps to 4, 4 maps to 5 and 5 maps to 1. And remaining 3 maps to 3 itself.

2.2 Multiplying permutations

Because the operation in an arbitrary group is frequently called multiplication, it is common to refer to the composition of permutations as multiplication and write $\pi_2\pi_1$ instead of $\pi_2 \circ \pi_1$. The important thing to remember here, however, is that the operation is simply function composition. For example, if $\pi_1 = (1234)$, and $\pi_2 = (12)(34)$, then observe that $\pi_3 = \pi_2\pi_1 = (1)(24)(3)$. Here, $\pi_4 = \pi_1\pi_2 = (13)(2)(4)$ and hence $\pi_2\pi_1 \neq \pi_1\pi_2$.

We conclude this section with one more example. Let's find $[(123456)][(165432)]$, where we've written the two permutations being multiplied inside brackets. Here, the product is $(1)(2)(3)(4)(5)(6)$, which is better known as the identity permutation. Thus, (123456) and (165432) are inverses.

In the next section, we will state the results for arbitrary groups, but you may safely replace *group* by *permutation group* without losing any understanding required for the remainder of the chapter.

3 Burnside's Lemma

Burnside's lemma relates the number of equivalence classes of the action of a group on a finite set to the number of elements of the set fixed by the elements of the group. Before stating and proving it, we need some notation and a proposition. If a group G acts on a finite set $\mathcal{C}$, let $\sim$ be the equivalence relation induced by this action. (The action of $\pi \in G$ on $\mathcal{C}$ will be denoted π^*.) Denote the equivalence class containing $C \in \mathcal{C}$ by $< C >$. For $\pi \in G$, let $fix_{\mathcal{C}}(\pi) = \{C \in \mathcal{C} : \pi^*(C) = C\}$, the set of colorings fixed by π. For $C \in \mathcal{C}$, let $stab_G(C)\{\pi \in G : \pi(C) = C\}$ be the *stabilizer* of C in G, the permutations in G that fix C.

Proposition 3.1. *Let a group G act on a finite set $\mathcal{C}$. Then for all $C \in \mathcal{C}$,*

$$\sum_{C' \in <C>} |stab_G(C')| = |G|.$$

Proof. Let $stab_G(C) = \{\pi_1, ..., \pi_k\}$ and $T(C, C') = \{\pi \in G : \pi^*(C) = C'\}$. (Note that $T(C, C) = stab_G(C)$.) Take $\pi \in T(C, C')$. Then $\pi \circ \pi_i \in T(C, C')$ for $1 \leq i \leq k$. Furthermore, if $\pi \circ \pi_i = \pi \circ \pi_j$, then $\pi^{-1} \circ \pi \circ \pi_i = \pi^{-1} \circ \pi \circ \pi_j$. Thus $\pi_i = \pi_j$ and $i = j$. If $\pi' \in T(C, C')$, then $\pi^{-1} \circ \pi' \in T(C, C)$. Thus, $\pi_{-1} \circ \pi = \pi_i$ for some i, and hence $\pi' = \pi \circ \pi_i$. Therefore $T(C, C') = \{\pi \circ \pi_1, \ldots, \pi \circ \pi_k\}$. Additionally, we observe that $T(C', C) = \{\pi^{-1} : T(C, C')\}$. Now for all $C' \in < C >$,

$$|stab_G(C')| = |T(C', C')| = |T(C', C)| = |T(C, C')| = |T(C, C)| = |stab_G(C)|$$

Therefore,

$$\sum_{C' \in <C>} |stab_G(C')| = \sum_{C' \in <C>} |T(C, C')|.$$

Now notice that each element of G appears in $T(C, C')$ for precisely one $C' \in < C >$, and the proposition follows. ☐

With Proposition established above, we are now prepared for Burnside's lemma.

Lemma 3.2. *(Burnside's Lemma). Let a group G act on a finite set $\mathcal{C}$. If N is the number of equivalence classes of $\mathcal{C}$ induced by this action, then*

$$N = \frac{1}{|G|} \sum_{\pi \in G} |fix_{\mathcal{C}}(\pi)|.$$

Proof. Let $X = \{(\pi, C) \in G \times \mathcal{C} : \pi(C) = C\}$. Notice that $\sum_{\pi \in G} |fix_C(\pi)| = |X|$, since each term in the sum counts how many ordered pairs of X have π in their first coordinate. Similarly, $\sum_{C \in \mathcal{C}} |stab_G(C)| = |X|$, with each term of this sum counting how many ordered pairs of X have C as their second coordinate. Thus, $\sum_{\pi \in G} |fix_C(\pi)| = \sum_{C \in \mathcal{C}} |stab_G(C)|$. Now note that the latter sum may be rewritten as

$$\sum_{\text{equivalence classes } <C>} (\sum_{C' \in <C>} |stab_G(C')|).$$

By Proposition stated above, the inner sum is $|G|$. Therefore, the total sum is $N.|G|$, so solving for N gives the desired equation. ☐

4 Pólya's Theorem

Before getting to the full version of Pólya's formula, we must develop a generating function as promised at the beginning of the note. Specifically, if π is a permutation of $[n]$ with j_k cycles of length k for $1 \leq k \leq n$, then the monomial associated to π is $x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$. Note that $j_1 + 2j_2 + 3j_3 + \ldots + nj_n = n$. We define the cycle index more formally below.

Definition 4.1. *Let G be a group of permutations. The cycle index is the polynomial with n variables $x_1, x_2, \ldots, x_n$*

$$Z_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x_1^{b_1}, x_2^{b_2}, \ldots, x_n^{b_n}$$

where the product $x_1^{b_1}, x_2^{b_2}, \ldots, x_n^{b_n}$ is formed for each $g \in G$ from its type $\{b_1, b_2, \ldots, b_n\}$ where b_i is the number i-cycles in the permutation representation.

For example, the permutation single clockwise rotation (1234) is associated with the monomial x_4^1 since it consists of a single cycle of length 4. The permutation (13)(24) has two cycles of length 2, and thus its monomial is x_2^2. For $p = (14)(2)(3)$, we have two 1-cycles and one 2-cycle, yielding the monomial $x_1^2 x_2^1$. Recall that Burnside's Lemma states that the number of colorings fixed by the action of a group can be obtained by adding up the number fixed by each permutation and dividing by the number of permutations in the group. If we do that instead for the monomials arising from the permutations in a permutation group G in which every cycle of every permutation has at most n entries, we obtain a polynomial known as the *cycle index* $P_G(x_1, x_2, \ldots, x_n)$. Hopefully the power of the cycle index to count colorings that are distinct when symmetries are considered is becoming apparent. For our example, identical permutation is (1)(2)(3)(4) having monomial x_1^4. Single clockwise rotation (1234) is associated with the monomial x_4^1 and similarly, two clockwise rotation (12)(34) is associated with x_2^2, three clockwise rotation (1432) is associated with x_4^1, flip on vertical line (12)(34) is associated with x_2^2, flip on horizontal line (14)(23) is associated with x_2^2, flip on positive slop diagonal (14)(2)(3) is associated with $x_1^2 x_2^1$ and flip on negative slop diagonal (1)(24)(3) is associated with $x_1^2 x_2^1$. We find $P_D(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2 x_2^1 + 3x_2^2 + 2x_4^1)$.

To find the number of distinct 2-colorings of the square, we thus let $x_i = 2$ for all i and obtain $P_D(2, 2, 2, 2) = 6$ as before. Notice, however, that we have something more powerful than Burnside's lemma here. We may substitute any positive integer m for each x_i to find out how many nonequivalent m-colorings of the square exist. We no longer have to analyze how many colorings each permutation fixes. For instance, $P_D(3, 3, 3, 3) = 21$, meaning that 21 of the 81 colorings of the vertices of the square using three colors are distinct.

However, we still haven't seen the full power of Pólya's technique. From the cycle index alone, we can determine how many colorings of the vertices of the square are distinct. However, what if we want to know how many of them have two white vertices and two gold vertices? This is where Pólya's enumeration formula truly plays the role of a generating function. Let's again consider the cycle index for the dihedral group D_8: $P_D(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2 x_2^1 + 3x_2^2 + 2x_4^1)$.

Instead of substituting integers for the x_i, let's consider what happens if we substitute something that allows us to track the colors used. Since x_1 represents a cycle of length 1 in a permutation, the choice of white or red for the vertex in such a cycle amounts to a single vertex receiving that color. What happens if we substitute $w + r$ for x_1? The first term in P_D corresponds to the identity permutation i, which fixes all colorings of the square. Letting $x_1 = w + r$ in this term gives $(w + r)^4 = r^4 + 4r^3 w + 6r^2 w^2 + 4rw^3 + w^4$, which tells us that i fixes one coloring with four red vertices, four colorings with three red vertices and one white vertex, six colorings with two red vertices and two white vertices, four colorings with one red vertex and three white vertices, and one coloring with four white vertices. Let's continue establishing a pattern here by considering the variable x_2. It represents the cycles of length 2 in a permutation. Such a cycle must be colored uniformly white or red to be fixed by the permutation. Thus, choosing white or red for the vertices in that cycle results in two white vertices or two red vertices in the coloring. Since this happens for every cycle of length 2, we want to substitute $w^2 + r^2$ for x_2 in the cycle index. The $x_1^2 x_2^1$ terms in PD are associated with the flips on positive and negative diagonal. Letting $x_1 = w + r$ and $x_2 = w^2 + r^2$, we find $x_1^2 x_2^1 = r^4 + 2r^3w + 2r^2w^2 + 2rw^3 + w^4$, from which we are able to deduce that positive and negative diagonal each fix one coloring with four red vertices, two colorings with three g vertices and one white vertex, and so on. Comparing this with Table described above shows that the generating function is right on.

By now the pattern is becoming apparent. If we substitute $w^i + r^i$ for x_i in the cycle index for each i, we then keep track of how many vertices are colored white and how many are colored red. The simplification of the cycle index in this case is then a generating function in which the coefficient on $r^s w^t$ is the number of distinct colorings of the vertices of the square with s vertices colored red and t vertices colored white. Doing this and simplifying gives $P_D(w + r, w^2 + r^2, w^3 + r^3, w^4 + r^4) = r^4 + r^3w + 2r^2w^2 + rw^3 + w^4$.

From this we find one coloring with all vertices red, one coloring with all vertices white, one coloring with three red vertices and one white vertex, one coloring with one red vertex and three white vertices, and two colorings with two vertices of each color. As with the other results we've discovered in this note, this property of the cycle index holds up beyond the case of coloring the vertices of the square with two colors. The full version is Pólya's enumeration theorem:

Theorem 4.2. *(Pólya's Enumeration Theorem) Let X and Y be finite sets, with $|X| = n$. Let G be a group acting on X and let Z_G be the cycle index polynomial. If w is the weight function on Y, then the configuration generating function is given by:*

$$Z_G\Big(\sum_{y\in Y} w(y), \sum_{y\in Y} w(y)^2, \ldots, \sum_{y\in Y} w(y)^n\Big).$$

The special case of assigning $w(y) = 1$ for all $y \in Y$ gives the total number of configurations: $Z_G(|Y|, |Y|, \ldots, |Y|)$.

Before proving the theorem we will prove the following lemma.

Lemma 4.3. *Let X be a disjoint union of sets $X_1, X_2, \ldots, X_n$. If $S \subset Y^X$ is the set of functions f which are constant on each X_i, then*

$$\sum_{f\in S} W(f) = \prod_{i=1}^{n} \sum_{y\in Y} w(y)^{|X_i|}$$

Proof. Suppose $Y = \{y_1, ..., y_m\}$. We begin by expanding the product

$$\prod_{i=1}^{k} \sum_{y\in Y} w(y)^{|X_i|}$$

$$= \big(w(y_1)^{|X_1|} + \ldots + w(y_m)^{|X_1|}\big)\big(w(y_1)^{|X_2|} + \ldots + w(y_m)^{|X_2|}\big) \ldots \big(w(y_1)^{|X_n|} + \ldots + w(y_m)^{|X_n|}\big).$$

Multiplying out the RHS of the above equation gives a sum

$$\big(w(y_1)^{|X_1|}w(y_1)^{|X_2|}\dots w(y_1)^{|X_n|}\big)+\big(w(y_1)^{|X_1|}w(y_1)^{|X_2|}\dots w(y_2)^{|X_n|}\big)\dots\big(w(y_m)^{|X_1|}w(y_m)^{|X_2|}\dots w(y_m)^{|X_n|}\big).$$

Since f is constant in each X_i iff $f \in S$, each term in this sum corresponds to a single function $f \in S$. Thus the sum can be written as

$$\sum_{f\in S} w(f(x_1))^{|X_1|}w(f(x_2))^{|X_2|}\dots w(f(x_n))^{|X_n|}$$

where $x_i \in X_i$ for each $i \in \{1, \dots, n\}$. Finally we notice that this can also be written as

$$\sum_{f\in S}\Big(\prod_{x\in X_1} w(f(x))\Big)\Big(\prod_{x\in X_2} w(f(x))\Big)\dots\Big(\prod_{x\in X_n} w(f(x))\Big)$$

$$= \sum_{f\in S}\prod_{x\in X} w(f(x))$$

$$= \sum_{f\in S} W(f)$$

Thus the lemma holds. □

Proof. (Pólya's Enumeration Theorem) Let $S_\omega \subseteq Y^X$ be the set of functions that have weight ω. Let $\phi_\omega(g) = \{f : X \to Y | f = f_g, W(f) = \omega\}$. By Burnside's lemma the number of configurations in S_ω is given by

$$|S_\omega| = \frac{1}{|G|}\sum_{g\in G} |\phi_\omega(g)|. \tag{1}$$

Thus multiplying by ω and summing over all values of ω gives the Configuration generating function:

$$\sum_{c\in C} W(c) = \sum_{\omega} \omega|S_\omega| = \frac{1}{|G|}\sum_{\omega}\sum_{g\in G}\omega|\phi_\omega(g)|. \tag{2}$$

Letting $\phi(g)$ denote the set $\{f : X \to Y | f = f_g\}$, we have that

$$\sum_{\omega}\omega|\phi_\omega(g)| = \sum_{f\in\phi(g)} W(f). \tag{3}$$

Since the double summation in equation() is taken over finite sets, the order of the summation may be switched. Thus switching the order of summation and combining with equation() gives the configuration generating function

$$\sum_{c\in C} W(c) = \frac{1}{|G|}\sum_{g\in G}\omega|\phi_\omega(g)|\sum_{f\in\phi(g)} W(f). \tag{4}$$

Every $g \in G$ permutes the set X. Therefore each g splits X into a disjoint union of m cycles $X_1, X_2, \dots, X_m$, where $m \le n$. If $f \in S$, then $f = fg = fg^2 = \dots$, i.e. f is constant on each X_i. If f is constant on each X_i, then $f = fg$, and hence $f \in S$. Now applying above lemma gives

$$\sum_{f \in \phi(g)} W(f) = \prod_{i=1}^{m} \sum_{y \in Y} w(y)^{|X_i|} = \Big(\sum_{y \in Y} w(y)^{|X_1|}\Big)\Big(\sum_{y \in Y} w(y)^{|X_2|}\Big) \cdots \Big(\sum_{y \in Y} w(y)^{|X_m|}\Big). \quad (5)$$

If the cycle type of g is $\{b_1, b_2, \ldots, b_n\}$, then in $|X_1|, |X_2|, \ldots, |X_m|$, the number i occurs b_i times. Thus we can write equation (5) as

$$\sum_{f \in \phi(g)} W(f) = \Big(\sum_{y \in Y} w(y)\Big)^{b_1} \Big(\sum_{y \in Y} w(y)^2\Big)^{b_2} \cdots \Big(\sum_{y \in Y} w(y)^n\Big)^{b_n}. \quad (6)$$

Substituting equation (6) into equation (4) gives the configuration generating function:

$$\frac{1}{|G|} \sum_{g \in G} \left(\Big(\sum_{y \in Y} w(y)\Big)^{b_1} \Big(\sum_{y \in Y} w(y)^2\Big)^{b_2} \cdots \Big(\sum_{y \in Y} w(y)^n\Big)^{b_n} \right) \quad (7)$$

Now we simply notice that equation (7) is precisely the cycle index

$$Z_G\Big(\sum_{y \in Y} w(y), \sum_{y \in Y} w(y)^2, \ldots, \sum_{y \in Y} w(y)^n\Big)$$

This proves Pólya's Enumeration Theorem. □

5 Applications

The section is concluded with some of Pólya's examples.

Example 5.1. *Given a set of black and white beads, how many distinct five beaded necklaces can be made if one allows for rotation of the necklaces? How many of them contain two black beads and three white beads?*

We let X be a five beaded necklace, and Y be the set containing the two colors black and white. The relevant permutation group G is the cyclic group C_5. We assign weights to the two colors as follows: $w(black) = B$ and $w(white) = W$. Now applying PET gives the CGF:

$$F(B, W) = Z_G(B + W, B^2 + W^2, \ldots, B^5 + W^5)$$
$$= \frac{1}{5}((B + W)^5 + 4(B^5 + W^5)) = B^5 + B^4W + 2B^3W^2 + 2B^2W^3 + BW^4 + W^5$$

To find the number of distinct necklaces which contain two black beads and three white beads we simply read off from the CGF the coefficient of the term B^2W^3, which is 2. To find the total number necklaces, we assign weights w(white) = w(black) = 1, which gives F(1,1) = 1+1+2+2+1+1 = 8.

Example 5.2. *Assume that two necklaces are identical if one can be formed from the other by rotations and/or reflections. We find the general formula for the number of necklaces that can be made with k_1 black beads and k_2 white beads, where $k_1 + k_2$ is an odd prime and $k_1, k_2 \neq 0$.*

First we let k_1 and k_2 be positive integers such that $k_1 + k_2 = n$, where n is an odd prime. Since n is odd, we can assume without loss of generality that k_1 is even. We assign weights B and W to the black and white beads respectively. Now the relevant group acting on the necklace is the dihedral group and since n is odd, we find the cycle index to be:

$$Z_{D_n}(x_1, x_2, \ldots, x_n)$$

$$= \frac{1}{2} Z_{C_n}(x_1, x_2, \ldots, x_n) + \frac{1}{2} x_1 x_2^{(n-1)/2}$$

Substituting the cycle index Z_{C_n} into above equation gives:

$$Z_{D_n}(x_1, x_2, \ldots, x_n)$$

$$= \frac{1}{2n} \sum_{d|n} \phi(d) x_d^n/d + \frac{1}{2} x_1 x_2^{(n-1)/2}$$

$$= \frac{1}{2n}(x_1^n + (n-1)x_n) + \frac{1}{2} x_1 x_2^{(n-1)/2}$$

Using the cycle index in the above equation, we now apply PET to get the configuration generating function:

$$F(B, W) = \frac{1}{2n}(B + W)^n + \frac{1}{2n}(n-1)(B^n + W^n) + \frac{1}{2}(B + W)(B^2 + W^2)^{(n-1)/2}$$

We need not expand the polynomials since we are only interested in the coefficient of the term $B^{k_1}W^{k_2}$. We look separately at the three summands of the above equation. First we notice that the coefficient of $B^{k_1}W^{k_2}$ in $\frac{1}{2n}(B + W)^n$ is given by the binomial theorem as $\frac{1}{2n}\binom{n}{k_1}$. The second summand $\frac{1}{2n}(n-1)(B^n+W^n)$ does not affect the coefficient of $B^{k_1}W^{k_2}$, since it is only adding to the coefficients of B^n and W^n. In the final summand, using the binomial theorem we see that $(B^2 + W^2)^{(n-1)/2}$ contributes $\binom{(n-1)/2}{k_1/2}$ to the coefficient of $B^{k_1}W^{k_2-1}$. Then multiplying through by $\frac{1}{2}(B + W)$ the coefficients of $B^{k_1}W^{k_2}$ becomes $\frac{1}{2}\binom{(n-1)/2}{k_1/2}$. Having now accounted for each of the summands, the coefficient of the $B^{k_1}W^{k_2}$ term is given by the sum:

$$\frac{1}{2n}\binom{n}{k_1} + \frac{1}{2}\binom{(n-1)/2}{k_1/2}$$

Hence the general formula for the number of necklaces that can be formed using k_1 black beads and k_2 white beads, where $k_1 + k_2$ is an odd prime, is given by:

$$\frac{1}{2(k_1 + k_2)}\binom{k_1 + k_2}{k_1} + \frac{1}{2}\binom{(k_1 + k_2 - 1)/2}{k_1/2}$$

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