Discrete Mathematics 17/04/2019

Lecture 21: Flow Network

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21.1 Introduction

In graph theory, a ***flow network*** (also known as a transportation network) is a directed graph where each edge has a capacity and there is a flow in each edge. The amount of flow on an edge cannot exceed the capacity of the edge. Often in operations research, a directed graph is called a ***network***, the vertices are called nodes and the edges are called arcs.

A ***flow*** must satisfy the restriction that the amount of flow into a node equals the amount of flow out of it, unless it is a source, which has only outgoing flow, or sink, which has only incoming flow. That is, there is no storage in the nodes excluding the source and the sink.

A ***network*** can be used to model traffic in a computer network, circulation with demands, fluids in pipes, currents in an electrical circuit, or anything similar in which something travels through a network of nodes.

21.2 Some Definitions

Definition 21.2.1. *(Network and Capacity Function) A* ***network*** *is a simple digraph $G = (V, E)$, where V is a set of vertices and E is a subset of $V \times V$ with a non-negative function $c : E \to \mathbb{R}^+$, called the* ***capacity function****.*

Without loss of generality, we may assume that if $(u, v) \in E$ then (v, u) is also a member of E, since if $(v, u) \notin E$ then we may add (v, u) to E and then set $c(v, u) = 0$.

Definition 21.2.2. *(Flow Network) If two nodes in G are distinguished, a source s and a sink t, then (G, c, s, t) is called a* ***flow network****.*

The capacity function $c(u, v)$ denotes the maximum flow of the edge between the two vertices u and v.

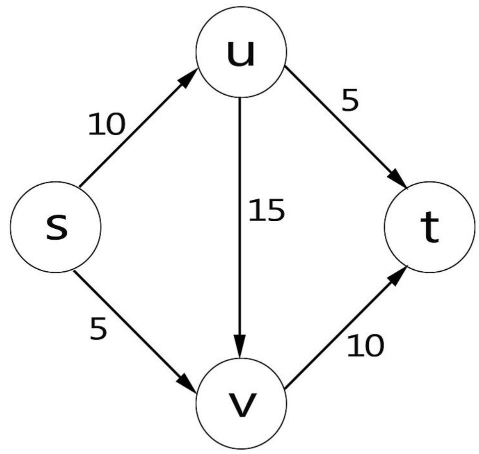

Figure 1

21.3 Flow Maximization Problem

At first let us consider a flow problem $G = (V, E)$ in Figure 1. The source and sink are represented by s and t. Now consider a digraph between them. u and v are two vertices of the digraph. The edges are shown with their respective capacities.

Flow Maximization Problem deals with how to control the flow of every edge so that the flow is maximized.There are several algorithms to do it. Here we will discuss about one of those algorithms.

21.3.1 Ford-Fulkerson Algorithm

Let $G(V, E)$ be a graph, and for each edge from u to v, let $c(u, v)$ be the capacity and $f(u, v)$ be the flow. We want to find the maximum flow from the source s to the sink t. After every step in the algorithm the following is maintained:

- $\forall (u, v) \in E : f(u, v) \leq c(u, v)$ the flow in each edge is obviously less than or equal to its capacity.
- $\forall (u, v) \in E : f(u, v) = -f(v, u)$. (Skew-symmetric Property)
- $\forall u \in V$ such that $u \neq s$ and $u \neq t$, we have:

  $$\sum_{w \in V} f(u, w) = 0.$$

  The net flow to a node is zero, except for the source, which "produces" flow, and the sink, which "receives" flow.
- The flow leaving from s must be equal to the flow arriving at t. Which means:

  $$\sum_{(s,u) \in E} f(s, u) = \sum_{(v,t) \in E} f(v, t).$$

ALGORITHM :

Algorithm 1. *C*

onsider a Network $G = (V, E)$ with flow capacity c, a source node s, and a sink node t. Now we want to calculate the path of maximum flow from s to t. The algorithm is as follows-

1.For all edges $(u, v) \in E$ first assign all the flow functions $f(u, v)$ as 0.

2.Now take a path from s to t and call it p, if for all edges $(u, v) \in p, c(u, v) > 0$.

3.Now find $c(p) = \min\{c(u, v) : (u, v) \in p\}$.

4.Now $\forall (u, v) \in p$ assign $f(v, u) = f(v, u) - c(p)$ and $f(u, v) = f(u, v) + c(p)$ and $c(u, v) = c(u, v) - c(p)$.

5.Return to step 2. Stop if you get no path p such that $c(u, v) > 0 \; \forall (u, v) \in p$.

end

21.4 Max-Flow Min-Cut Theorem

Before discussing the theorem let us first define a useful term.

Definition 21.4.1. *(Cut) Let (G, t, s, c) be a flow network. An $s - t$* ***cut*** *in G is a partition of V into two sets S and T such that:*

- $S \cup T = V$.
- $S \cap T = \phi$.
- $s \in S$ *and* $t \in T$.

Now look at the Figure 2. Here an example of ***cut*** $C(S, T)$ is $S = \{s, c, d\}$ and $T = \{a, b, t\}$.

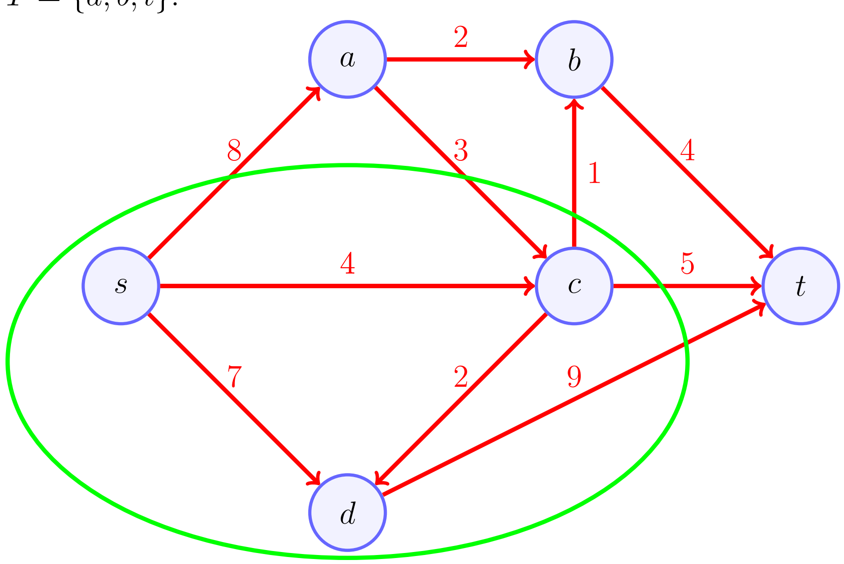

Figure 2.

Definition 21.4.2. *(Capacity of a Cut) The capacity of a cut $C(S,T)$, denoted $c(S,T)$ is the sum of the capacities of the edges (u,v) with $u \in S$ and $v \in T$.*

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v).$$

In the Figure 2, $c(S,T) = c(c,t) + c(c,b) + c(d,t) + c(s,a) = 23$.

We can define the flow of a cut in a similar manner. That is, given a cut $C(S,T)$ with capacity $c(S,T)$, and a flow f, the amount of flow will be the flow going from S to T minus whatever flow was sent back.

Definition 21.4.3. *(Flow of a Cut) The flow of a cut $C(S,T)$ denoted by $f(S,T)$ is:*

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u).$$

Theorem 21.4.4. *The maximum value of an $s-t$ flow is equal to the minimum capacity over all $s-t$ cuts.*

To prove this theorem we need to know some lemmas.

Lemma 21.4.5. *Let (G,s,t,c) be a flow network and $C(S,T)$ is an $s-t$ cut and f is a flow. Then:*

$$f(S,T) \leq c(S,T).$$

Proof. We know that for all (u,v) in the network

$$f(u,v) \leq c(u,v).$$

Now,

$$\begin{aligned} f(S,T) &= \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) \\ &\leq \sum_{u \in S} \sum_{v \in T} f(u,v) \\ &\leq \sum_{u \in S} \sum_{v \in T} c(u,v) \\ &= c(S,T). \end{aligned}$$

□

Lemma 21.4.6. *Let (G,s,t,c) be a flow network and $C(S,T)$ is an $s-t$ cut and f is a flow and $v \in T$. Then:*

$$f(S,T) = f(S \cup \{v\}, T \setminus \{v\}).$$

Proof. Let $C(S,T)$ be any *st* cut and $v \in T$. Remove v from T and place it in S, and now lets evaluate the flow of this new cut $C'(S',T')$ where $S' = S \cup \{v\}$ and $T' = T \setminus \{v\}$.

Let $In(v) = \{(u,v) \in E | u \in V\}$ denotes the incoming edges to v and $Out(v) = \{(v,w) \in E | w \in W\}$ denotes the outgoing edges from v.

From conservation of flow we know that:

$$\sum_{(u,v) \in In(v)} f(u,v) = \sum_{(v,w) \in Out(v)} f(v,w).$$

Now we partition the $In(v)$ and $Out(v)$ based on where the end point lies.

$$In_S(v) = \{(u,v) \in E | u \in S\}.$$

$$In_T(v) = \{(u,v) \in E | u \in T\}.$$

$$Out_S(v) = \{(v,w) \in E | u \in S\}.$$

$$Out_T(v) = \{(v,w) \in E | w \in T\}.$$

Lets evaluate the flow of this new cut now. Moving v into S will result in loosing vs contribution to the original cut capacity, but also in gaining the capacity of the outgoing edges from v. Thus:

$$\begin{aligned} f(S \cup \{v\}, T \setminus \{v\}) = f(S,T) &- \left(\sum_{(u,v) \in In_S(v)} f(u,v) + \sum_{(u,v) \in In_T(v)} f(u,v) \right) \\ &+ \left(\sum_{(v,w) \in Out_S(v)} f(v,w) + \sum_{(v,w) \in Out_T(v)} f(v,w) \right) \end{aligned}$$

$$= f(S,T) - \left(\sum_{(u,v) \in In(v)} f(u,v) - \sum_{(v,w) \in Out(v)} f(v,w) \right) = f(S,T).$$

This is because $In_S(v) + In_T(v) = In(v)$ and $Out_S(v) + Out_T(v) = Out(v)$. □

Now let us consider a cut $C(S,T) = S = \{s\}, T = V \setminus \{s\}$ and we apply lemma 21.4.6 on it. So we get:

$$f(\{s\}, V \setminus \{s\}) = f(\{s\} \cup S', V \setminus \{s\} \cup S') = f(S',T') \quad \forall S' \subset V \setminus \{s\}. \qquad (1)$$

Here $T' = V \setminus \{s\} \cup S'$. This says that the flow of any cut equals the flow of the first cut we started with, namely $(\{s\}, V \setminus \{s\})$.

Similarly $c(\{s\}, V \setminus \{s\})$ is just the maximum flow that can leave the source s, and we know the this flow is just the maximum possible flow of the network.

Definition 21.4.7. *(Value of Flow)*

$$val(f) = \sum_{(s,u)\in E} f(s,u) = f(\{s\}, V \setminus \{s\}). \tag{2}$$

Therefore using (1) and (2) we get the lemma below.

Lemma 21.4.8. *Let (G, s, t, c) be a flow network and $C(S,T)$ is an $s-t$ cut and f is a flow. Then:*

$$f(S,T) = val(f).$$

Corollary 21.4.9. *Let (G, s, t, c) be a flow network and $C(S,T)$ is an $s-t$ cut and f is a flow. Then:*

$$val(f) \le c(S,T).$$

Proof. (Max-Flow Min-Cut Theorem) Corollary 21.4.9 says that the capacity of any cut is at least the value of a flow on G. For any cut its capacity is minimized when its capacity value equals its value of flow (i.e. $c(S,T) = val(f)$). But this only happens when f itself is the maximum flow of the network! Therefore, in any flow network (G, s, t, c), the value of the maximum flow equals the capacity of the minimum cut in the network.

Hence, the proof is done. □