Discrete Mathematics 10/04/2019

Lecture 19: More on Graph Coloring

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1 Recalling some definitions

Definition 1.1. ***K-colouring***

Given a graph $G = (V, E)$ where V denote the vertex set and E the edge set,a k-colouring of G is a function $\psi : V \to \{1, 2, ..., k\}$ such that if $\{x, y\} \in E$,then $\psi(x) \neq \psi(y)$. If such a colouring exists the graph is called k-colourable. Colouring here is by default Vertex Colouring.

2 Chromatic Number and Chromatic polynomial

Definition 2.1. ***Chromatic Number***

The Chromatic Number of a Graph G, written as $\chi(G)$, is the minimum number of colours need to label the vertices so that adjacent vertices gets different colours.

Note : Given a Graph, it is *k-partite*(can be dissolved into k disjoint independent sets) if and only if it's Chromatic Number is at most k.

Definition 2.2. ***Chromatic Polynomial***

A *Chromatic Polynomial* denoted by $Q_G(x)$ of a Graph G is a polynomial in x, which counts the *no of ways* of colouring G with <u>at most</u> x colours.

3 Some basic results

3.1 $\chi(G)$ is the minimum integer k such that $Q_G(k) \neq 0$

Atlest k colour is required to label vertices of the Graph G. Hence $Q_G(k)$ has value atleast 1.

3.2 $\chi(G) \geq \omega(G)$, $\omega(G)$ denoting the size of the maximum clique in G

If a graph contains a clique of size k, then at least k colors are required to color just the clique. Thus, the chromatic number is at least k.

3.3 $\chi(I_n) = 1$, where I_n denotes an independent set of size n

An independent set can have all the vertices with same colour.

3.4 $\chi(K_n)$=n, K_n denotes clique of size n

K_n can be coloured in more than n ways.

3.5 $Q_{I_n}(x) = x^n$.

The no of ways of colouring n non-adjacent vertices is x for each vertex thus totalling x^n.

3.6 $Q_{K_n}(x) = x(x-1)(x-2)...(x-n+1)$.

All the vertices must have different colours. As a result, the first vertex has choice x, the next have choice $x-1$ and so on.

3.7 $\chi(G) \leq \Delta(G) + 1$, $\Delta(G)$ is the maximum degree in G.

One shows this by *greedy colouring*. One colors a vertex v_1 by atmost $\Delta(G)+1$ colors (1 for itself). If the next vertex v_2 is adjacent or non adjacent to v_1, even then no of coloring will remain $\Delta(G) + 1$.

3.8 $\chi(G) \geq \lceil n/\alpha(G) \rceil$, $\alpha(G)$ denotes the size of maximum independent set.

Proof: -We require $\chi(G)$ many colours; so lets partition the vertex set V into $\chi(G)$ many partitions, such that each partiton contains vertex with same colouring. As a result,we have P_1,P_2,....,$P_{\chi(G)}$ partitions. Therefore, total no of vertices:

$$\sum_{k=1}^{\chi(G)} |P_k| \leq \sum_{k=1}^{\chi(G)} \alpha(G) = \chi(G).\alpha(G) = n.$$

Therefore: $\chi(G) \geq (n/\alpha(G))$ completing the proof.

3.9 $\chi(G) = k$ iff G is k-partite.

Follows from the definition of k partite graphs: the graphs that can be divided into k disjoint maximal independent sets.

3.10 G is k-colourable iff G is k-partite

K-partite, as a result can be divided into k disjoint independent sets and is k-colourable.

Theorem 3.1. ***Two piece Theorem*** If G has two connected components G_1 and G_2, then

$$\mathbf{Q_G(x) = Q_{G_1(x)}.Q_{G_2(x)}}$$

Proof: This can be argued by basic product rule. Since G_1 and G_2 are connected components as a result these two are disjoint and exhausts to G. Thus one can choose the no of ways to colour component G_1 by $Q_{G_1}(x)$ and component G_2 by Q_{G_2} ways. Since these two components are disjoint; colouring for each vertex of G_1 corresponds to colouring of all the vertices of G_2. Thus product form applies and completes the proof.

4 Fundamental Reduction Theorem And its Application

Statement: Suppose E is an edge in a graph G connecting vertices $\{u$,$v\}$. G/E is the reduced graph obtained by removing the edge E, making the endpoints to be non adjacent. $G.E$ is the graph obtained by contracting the vertices along the edge E; thus reducing the vertex no by 1. Furthur assume the graph does not contain any self loop. Then,

$$\mathbf{Q_G(x) = Q_{G/E}(x) - Q_{G.E}(x)}$$

Proof:

- **Case 1**: $\{u, v\}$ is connected by a single edge E
  In that scenario, $Q_G(x)$ can be dissociated into 2 parts : one with the Graph having edge E removed, resulting in 2 non adjacent vertices, where each can be coloured x^2 ways. And the 2nd part consists the contraction of $\{u$,v$\}$ along the edge E, resulting in formation of 1 vertex contributing x, resulting in $x^2 - x$ which is the chromatic polynomial of the original graph.

- **Case 2**: $\{u, v\}$ is connected by more than 1 edge.
  Here $Q_{G/E}(x)$=$Q_G(x)$ due to the reason that the colouring of the edges $\{u, v\}$ is affected only by the adjacency of the two vertices. If edge E is removed,other edges remains fixed, as a result no of ways of colouring remains unchanged. $Q_{G.E}(x) = 0$.

Facts to notice here :

• In the statement of the theorem, we are considering those graphs, that are without loop. This can be explained by a simple example. Consider the above graph:

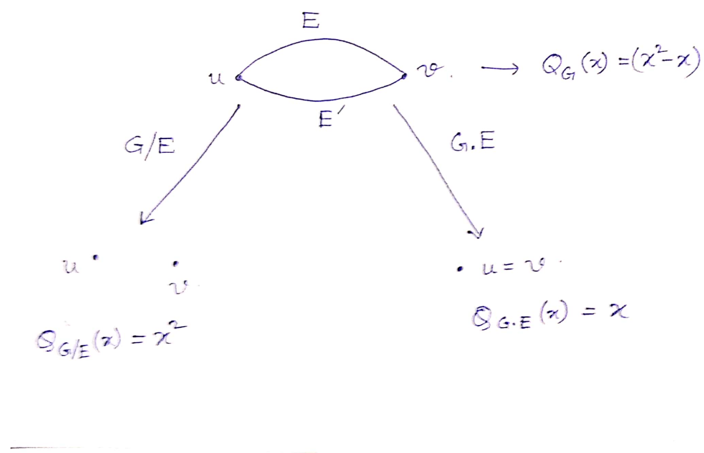

Figure 1

• We have two notions regarding the reduction of the Graph. Figure 1 follows the *first notion*. The notion is that if we have simple graph or multigraph we delete every edges so as to make a change in the colouring of the endpoints of the concerned edge.

The above example leads to formation of one graph (G/E) with vertices $\{u, v\}$ being non-adjacent and the other graph $(G.E)$ reducing it to one vertex. Chromatic polynomials for the respective graphs are x^2 and x. Subtracting the two results $x(x-1)$ which is equal to $Q_G(x)$.

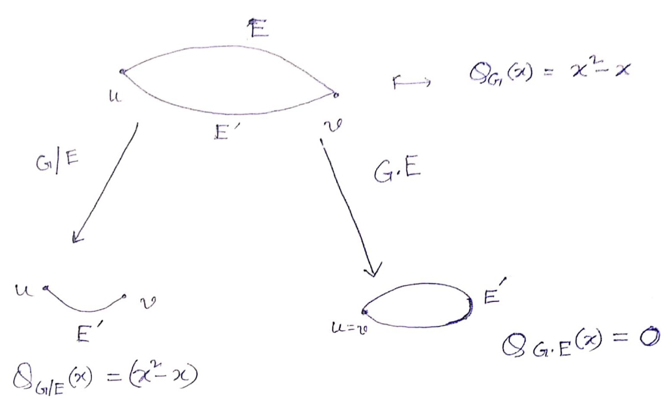

Figure 2

• For figure 2, the *2nd notion* says that if we have a simple graph, the contraction of the vertices $\{u, v\}$ along the single edge E leads to formation of a self loop with a single vertex u=v. Since we consider the colourability of a self loop to be zero, one gets $Q_{G/E}(x) = x(x-1)$, and equals to the chromatic polynomial of the graph G.

Theorem 4.1. ***Statement:*** *Consider a graph G=$\{V, E\}$, with $|V|$=n and $|E|$=m. Let*

$$Q_G(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_1 x + c_0$$

. Then,

- *The polynomial is monic. Coefficient of x^n i.e. c_n=1*
- *Constant term of the polynomial, i.e. c_0=0.*
- *$|c_{n-1}| = m$, the no of edges, and the coefficients alternate in sign.*

Proof *(Induction Aprroach:)*

We will proceed by induction on no of edges.

Base Case: $m = 0$

• Note that $Q_G(0) = 0$. Therefore, c_0=0. Now $m = 0$ implies G=I_n.

Therefore, $Q_G(x) = x^n$ implying $c_n = 1$,$c_0 = 0$ and $c_{n-1} = m = 0$. Done.

We impose the induction hypothesis and assume that the statement is true for all graphs with fewer than m edges *(Strong form of induction)*. Let M be one of the edges of G. By the Fundamental Reduction Theorem,

$$\mathbf{Q_G(x) = Q_{G/E}(x) - Q_{G.E}(x)}$$

• Now, since G/E has $m-1$ edges and n vertices, by the induction hypothesis, $Q_{G/E}(x)$ is monic of degree n. Since $G.E$ has $m-1$ edges and only $n-1$ vertices, $Q_{G.E}(x)$ is monic of degree $n-1$.

Upon subtraction of these polynomials, we see that there is no term in $Q_{G.E}(x)$ that can remove the x_n term in $Q_{G/E}(x)$; so $Q_G(x)$ will be monic of degree n.

• By induction hypothesis, assume coefficients alternate in sign and penultimate coefficient term = no of edges is true for all graphs on n vertices with fewer than $m-1$ edges. Therefore

$$Q_{G/E}(x) = x^n - (m-1)x^{n-1} + a_{n-2}x^{n-2} - a_{n-3}x^{n-3} + ...$$

$$Q_{G.E}(x) = x^{n-1} - (m-1)x^{n-2} + b_{n-3}x^{n-3} - b_{n-4}x^{n-4} + ...$$

Upon Subtracting we get:

$$Q_G(x) = x^n - mx^{n-1} + (a_{n-2} + (m-1))x^{n-2} - (a_{n-3} + b_{n-3})x^{n-3} + ...$$

Thus the signs alternate and none of the coefficients are zero. And $c_{n-1} = m$ is true for all graphs with n vertices and m edges.Thus completes the proof.

Proof (Principle of Inclusion and exclusion):

Let $P_1, P_2, ...P_m$ be the properties where P_i denotes the set of way of vertex colouring, such that the two vertices joining Edge i have the same colour.

Need to find cardinality of the following event: $P_1^c \cap P_2^c \cap P_3^c \cap ..... \cap P_m^c$.

Fix the no of colours to be x. By the principle of inclusion and exclusion:

$$N(P_1^c, P_2^c, ..., P_m^c) = N - \sum_{1 \leq i \leq m} N(P_i) + \sum_{1 \leq i \leq j \leq m} N(P_i P_j) + ... + (-1)^m N(P_1, P_2, ..., P_m)$$

Now, N=x^n, since one colours every vertex in x possible ways.

$N(P_i)$ consists of one edge i such that it's endpoints are coloured the same. For this case the two vertices are considered paired and can be coloured in x ways. The rest $n-2$ vertices can be coloured in x^{n-1} ways. Therefore, $N(P_i) = x.x^{n-2} = x^{n-1}$

So,

$$\sum_{1 \leq i \leq m} N(P_i) = m.x^{n-1}$$

Similarly, for the event $P_i \cap P_j$ we have two edges with their vertices having same colouring; which is possible for two cases,

- *Case 1 :* The two edges E_i and E_j are joined by a common vertex. In that case three vertices will have the same colouring joined by two edges E_i and E_j. Thus no of ways to colour this 3 vertices is x and the rest $n-3$ vertices can be coloured in x^{n-3} ways. Total no of ways to do this is x^{n-2} ways.
- *Case 2 :* The two edges are disjoined. Thus effectively two vertices in pair are needed to be coloured same, which can be done in x^2 ways. Rest vertices are coloured in x^{n-4} ways. Total no of ways will be x^{n-2} Thus

$$\sum_{1 \leq i \leq j \leq m} N(P_i P_j) = \sum_{1 \leq i \leq j \leq m} x^{n-1}$$

which is a positive term since one applies the principle of counting here.

Therefore

$$N(P_1^c, P_2^c, ..., P_m^c) = Q_G(x) = x^n - mx^{n-1} + a_{n-2}x^{n-2} + ... + (-1)^m a_1 x$$

where $a_1, a_2, ..., a_{n-2}$ denotes the counts which are positive integers, which is the required expression of the chromatic polynomial. Proved .

5 Map Colouring and Colour Theorems

Result 5.1. Perhaps the most famous problem in graph theory concerns map coloring: Given a map of some countries, how many colors are required to color the map so that countries sharing a border get different colors? It was long conjectured that any map could be colored with four colors, and this was finally proved in 1976.

Recall from Lecture 18 (Planar Graphs), we discussed about duality of graphs where we stated and proved a theorem. Let's recall the theorem.

Theorem 5.2. *A graph G is planar iff its dual G^* is planar. (We defined the dual graph G^* of a plane graph G as a graph whose vertices correspond to the faces of G.)*

We are practically using the idea of the above stated theorem in the following example:

Here is an example of a small map, colored with four colors:

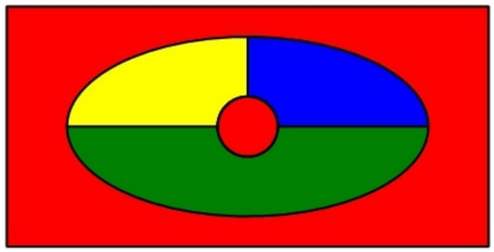

Figure 3: Map Colouring

Typically this problem is turned into a graph theory problem. Suppose we add to each country a capital, and connect capitals across common boundaries. Coloring the capitals so that no two connected capitals share a color is clearly the same problem. For the previous map:

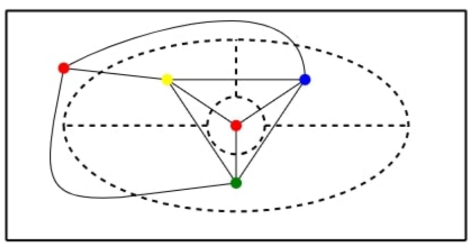

Figure 4: Vertex Colouring

Theorem 5.3. ***The Six Colour Theorem*** (Statement): *For a connected planar simple graph G, the vertices in G can be coloured with 6 or fewer colours for a good 6 (or less) colouring of G, that is, a function f exists $f : V \to \{1, 2, ..., k\}$ with $1 \leq k \leq 6$, such that if $\{x, y\} \in E$,then $f(x) \neq f(y)$.*

Proof: Let $S(n)$ be the statement that for a connected planar simple graph G, the vertices in G can be coloured with 6 or fewer colours for a good colouring of G.

Induction Base Step : For $1 \leq n \leq 6$, this is trivially true. A graph on 1 vertex can easily be coloured with just 1 colour, while a graph with 6 vertices can easily be coloured with just 6 colours for a good colouring (recall that we restrict ourselves to simple graphs).

Induction Step : Suppose that for all $k \geq 2$, $S(k-1)$ is true. That is, for all connected planar simple graphs on $k1$ vertices, we can obtain a good colouring of the vertices in G with 6 or fewer colours. We want to verify that $S(k)$ is true (that for all connected planar simple graphs on k vertices, we can obtain a good colour of the vertices in G with 6 or fewer colours still).

Now let G be a connected planar simple graph on k vertices. Recall that a connected planar simple graph G contains a vertex of degree 5 or less. Suppose the vertex v has deg(v)=5.

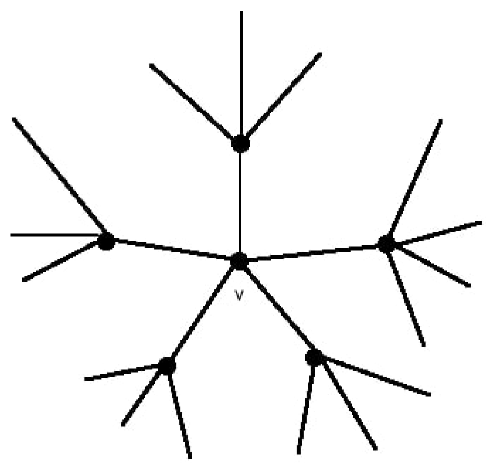

Figure 5

Now suppose that we remove vertex v and all of the edges incident with v. This graph now has less than k vertices, and by our induction hypothesis, we know this resulting graph can be coloured with 6 or fewer colours.

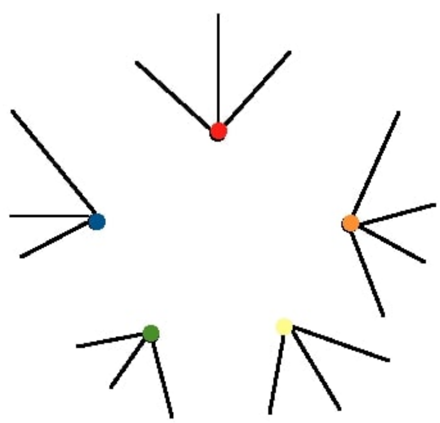

Figure 6

Adding vertex v back, we know that the neighbourhood of v contains 5 members. Hence if we use the 6th colour for vertex v, our proof is complete.

Hence $S(k1)$ implies $S(k)$. By the principle of mathematical induction, for $n \geq 1$, $S(n)$ is true.

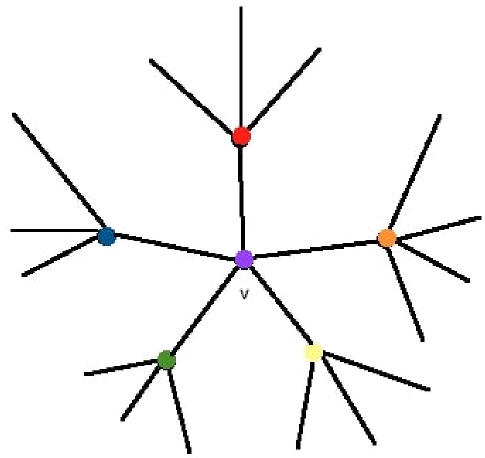

Figure 7

Theorem 5.4. ***The Five Colour Theorem*** (Statement) : *Every planar graph can be 5-colored.*

Proof We will again do this by induction on the number of vertices.

Base Case: For $1 \leq n \leq 5$, this is trivially true.The simplest connected planar graph consists of a single vertex. Pick a color for that vertex. So We are done.

Induction Hypothesis: Assume $k \geq 1$, and assume that every planar graph with $k-1$ or fewer vertices can be 5-colored. Now consider a planar graph with k vertices. From above, we know that the graph has a vertex of degree 5 or fewer. Remove that vertex (and all edges connected to it). By the induction hypothesis, we can 5-color the remaining graph. Put the vertex (and edges) back in. We have a graph with every vertex colored (without conflicts) except for the one.

If the vertex has degree less than 5, or if it has degree 5 and only 4 or fewer colors are used for vertices connected to it, we can pick an available color for it, and we are done.

If the vertex has degree 5, and all 5 colors are connected to it, we have a little more work to do. In this case, using numbers 1 through 5 to represent colors, we label the vertices adjacent to the special (degree 5) vertex 1 through 5 (in order).

And one can show this will leave color 1 available to color the special vertex,and we are done.

On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the same subgraph process with vertices colored 2 and 4 adjacent to the special vertex and complete the proof.

Thus, we will be able to color the entire planar graph with 5 colors, and the induction is done.

Theorem 5.5. ***The Four Colour Theorem*** (Statement): *Every planar graph can be 4-colored*

It turns out that it is actually a theorem that 4 colors are enough for any planar graph . The proof of that fact is significantly more difficult, and has only been done with the aid of exhaustive computer analysis of many special cases.

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