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Lecture 18: Planar Graphs

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18.1 Introduction

Three sworn enemies A, B, C live in houses in the woods. We must cut paths so that each has a path to each of three utilities, which can be considered gas, water and electricity. In order to avoid confrontations, we don't want any of the paths to cross. Can this be done? This asks whether $K_{3,3}$ can be drawn in the plane without edge crossings; which we will proof cannot be done.

In this following section, we shall be studying this question of whether a graph can be drawn in the plane without edges crossing. In particular, we shall answer the houses-and-utilities problem.

Definition 18.1 (Planar Graph)**.** A graph is called planar if it can be drawn on a plane (with vertices as points and edges as continuous curves) such that no two edges cross each other.

Such a drawing is called a "Planar Embedding" of the graph.

Example.(i): The following graph G is a planar graph.

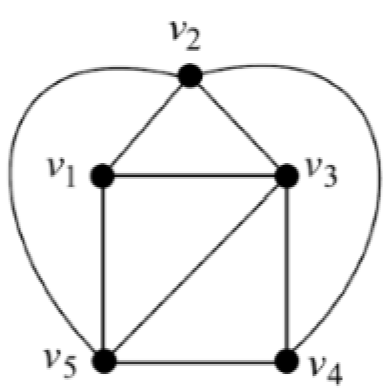

Fig. 18.1. Planar Graph G

Example.(ii): The following two graphs both are K_4 but the first one is a non-planar embedding while the second one is planar embedding of K_4.

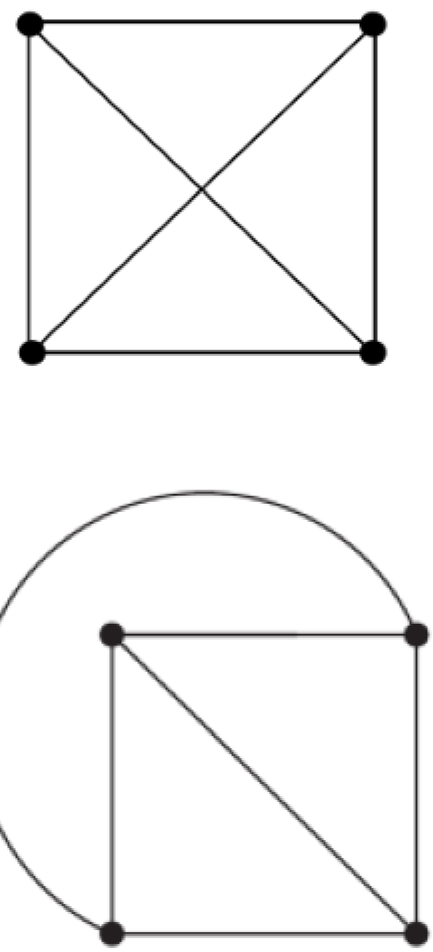

Fig 18.2. Example of graph in both planar and nonplanar embedding

Example.(iii): The following graphs are clearly two non-planar graphs.

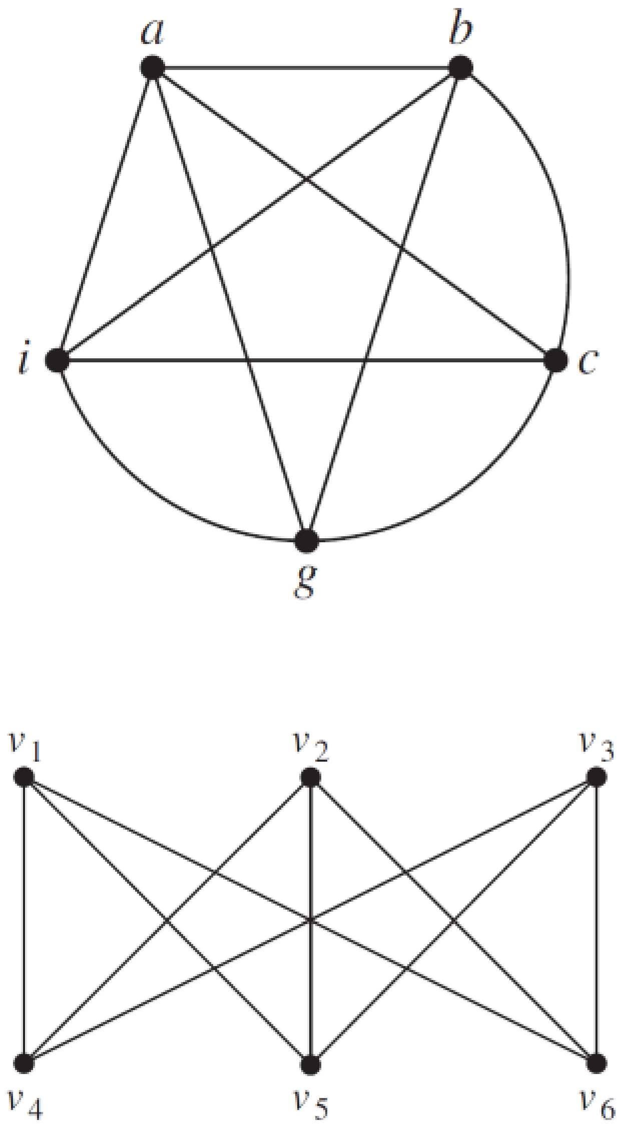

Fig.18.3. Two nonplanar graphs

Consider the two graphs in Example.(iii). The two given figures are non-planar embedding of these two graphs, but can one draw planar embedding of them? In fact, there exists no planar embedding of them. To prove that there exists a planar embedding it is sufficient to provide with a planar embedded figure, though it is a little more complicated to argue that there exists no planar embedded configuration.

18.2 Kuratowski's Two Graphs

The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are called Kuratowski's graphs, after the polish mathematician Kazimierz Kuratowski, who found that K_5 and $K_{3,3}$ are nonplanar.
To prove that these two graphs cannot be drawn in a plane, we shall essentially try to do the same and eventually end up getting two intersecting edges. In the procedure, we use an intuitive yet a topologically significant theorem, namely, Jordan Curve Theorem.

Theorem 18.2. *(Jordan Curve Theorem).*
(i) Any closed non-self-intersecting continuous curve J partitions the plane into 3 parts namely, interior of J ($intJ$), exterior of J ($extJ$) and J.

(ii) If J is a closed non-self-intersecting continuous curve, $s \in intJ$ and $t \in extJ$, then any continuous curve J' from s to t contains a point of J i.e. J' intersects J.

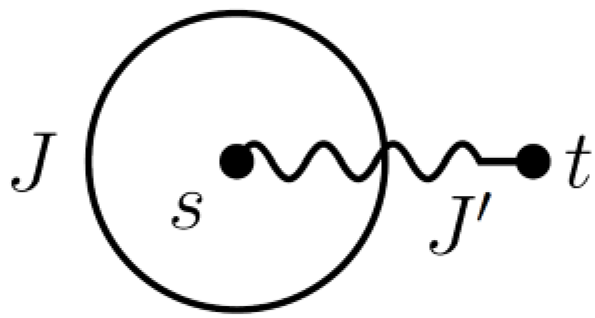

Fig.18.4. Jordan Curve Theorem

Since, its proof uses many topological aspects, we skip the proof here. But, the idea of this theorem will be used in the next two proofs. Let us move on to the theorems regarding the nonplanarity of the Kuratowski's graphs.

Theorem 18.3. The complete graph K_5 with five vertices is nonplanar.

Proof. Let the five vertices in the complete graph be named v_1, v_2, v_3, v_4, v_5. Since in a complete graph every vertex is joined to every other vertex by means of an edge, there is a cycle $v_1v_2v_3v_4v_5v_1$ that is a pentagon. This pentagon divides the plane of the paper in two regions, one inside and the other outside, Figure 18.5(a). Since vertex v_1 is to be connected to v_3 by means of an edge,this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Suppose we choose to draw the line from v_1 to v_3 inside the pentagon, Figure 18.5(b). In case we choose outside, we end with the same argument. Now we have to draw an edge from v_2 to v_4 and another from v_2 to v_5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon, Figure 18.5(c).The edge connecting v_3 and v_5 cannot be drawn outside the pentagon without crossing the edge between v_2 and v_4. Therefore v_3 and v_5 have to be connected with an edge inside the pentagon, Figure 18.5(d).

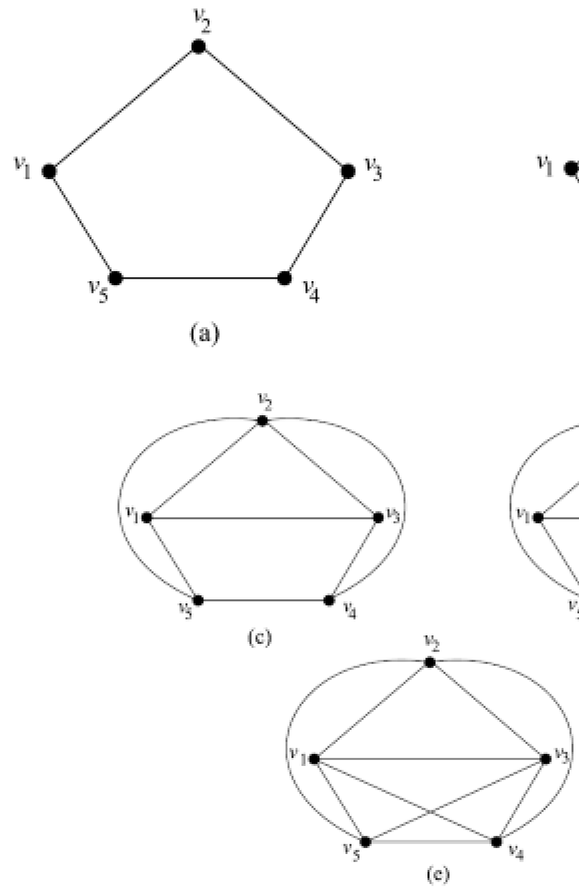

Fig. 18.5. Drawing K_5

Now,we have to draw an edge between v_1 and v_4 and this cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane.

□

Theorem 18.4. The complete bipartite graph $K_{3,3}$ is nonplanar.

Proof. The complete bipartite graph has six vertices and nine edges. Let the vertices be $u_1, u_2, u_3, v_1, v_2, v_3$. We have edges from every u_i to each v_i, $1 \leq i \leq 3$. First we take the edges from u_1 to each of v_1, v_2 and v_3. Then we take the edges between u_2 to each v_1, v_2 and v_3, Figure 18.6(a).

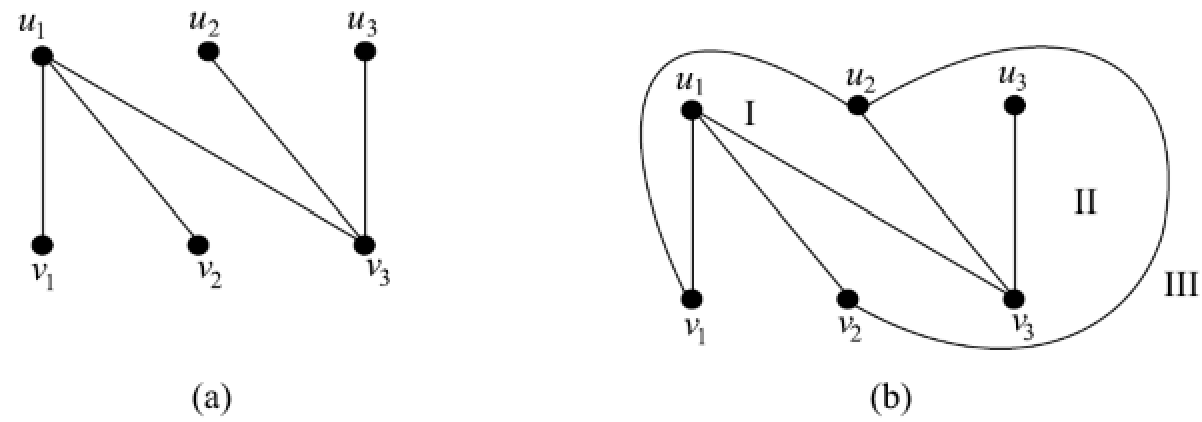

Fig. 18.6. Drawing $K_{3,3}$

Thus we get three regions namely I, II and III. Finally we have to draw the edges between u_3 to each v_1, v_2 and v_3. We can draw the edge between u_3 and v_3 inside the region II without any crossover, Figure 18.6(b). But the edges between u_3 and v_1, and u_3 and v_2 drawn in any region have a crossover with the previous edges. Thus the graph cannot be embedded in a plane. Hence $K_{3,3}$ is nonplanar.

□

We observe that the two graphs K_5 and $K_{3,3}$ have the following common properties.

1. Both are regular i.e. all the vertices have same degree.

2. Both are nonplanar.

3.Removal of one edge or a vertex makes each a planar graph.

4.K_5 is a nonplanar graph with the smallest number of vertices, and $K_{3,3}$ is the nonplanar graph with smallest number of edges.

Thus both are the simplest nonplanar graphs.

18.3 Faces or Regions

Definition 18.5. An **open set** in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every $p \in U$, all points within some small distance from p belong to U. A **region** is an open set U that contains a polygonal u, v-curve for for every pair $u, v \in U$. The **faces** of a plane graph are the maximal regions of the plane that contain no point used in the embedding.

A plane representation of a graph divides the plane into regions (also called windows, faces or meshes). A region is characterised by the set of edges (or the set of vertices) forming its boundary. We note that a region is not defined in a nonplanar graph, or even in a planar graph not embedded in a plane. Thus a region is a property of the specific plane representation of a graph and not an abstract graph.

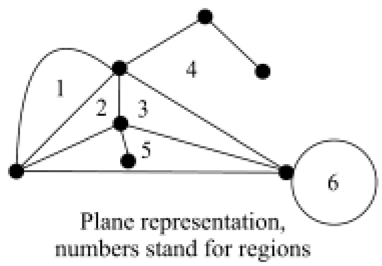

Fig. 18.7. Faces or Regions of a Graph

18.3.1 Dual Graphs

Definition 18.6. The dual graph G^* of a plane graph G is a graph whose vertices correspond to the faces of G. The edges of G^* correspond to the edges of G as follows: if e is an edge of G with face X on one side and face Y on the other side, then the endpoints of the dual edge $e^* \in E(G^*)$ are the vertices x, y of G^* that represent the faces of X, Y of G. The order in the plane of the edges incident to $x \in V(G^*)$ is the order of the edges bounding the face X of G in a walk around its boundary.

For example, consider Figure 18.8.

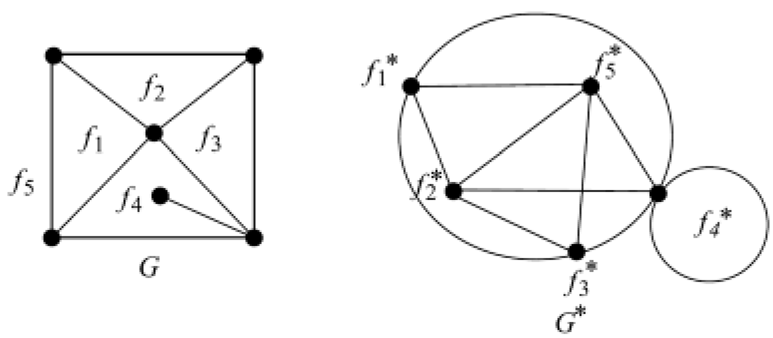

Fig. 18.8. Dual Graphs

Remark 18.7. Note that, **dual of dual of a graph G is G itself** i.e. $(G^*)^* = G$.

Theorem 18.8. A graph G is planar iff its dual G^* is planar.

Proof. Since, we have that $(G^*)^* = G$, it suffices to prove that G^* would be planar whenever G is planar. For the reverse direction, the exact same proof would hold as given G^* is planar, $(G^*)^*$ which is G would be planar. Let us prove: *The dual G^* of a plane graph is planar.*
Let G be a planar graph and let G^* be the dual of G. The following construction of G^* essentially proves our claim.
Place each vertex f_k^* of G^* inside the corresponding f_k of G. If the edge e_i lies on the boundary of two regions f_j and f_k of G, join the two vertices f_j^* and f_k^* by the edge e_i^*, drawing so that it crosses the edge e exactly once and crosses no other edge of G (Fig. 18.9).

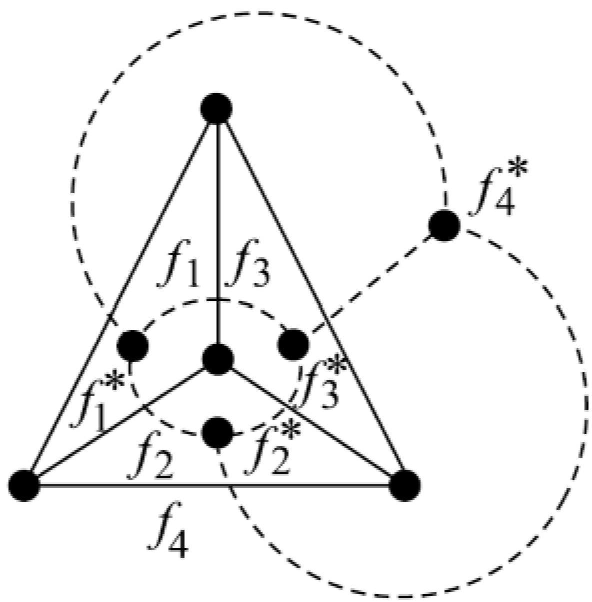

Fig. 18.9. Drawing Dual Graph

□

Remark 18.9. Clearly, there is one-one correspondence between the edges of planar graph G and its dual G^* with one edge of G^* intersecting one edge of G.

1. An edge forming a self-loop in G gives a pendant edge in G^* (An edge incident on a vertex of degree 1).

2. A pendant edge in G gives a self-loop in G^*.

3. Edges that are in series in G produce parallel edges in G^*.

4. Parallel edges in G produce edges in series in G^*.

5. The number of edges forming the boundary of a region f_i in G is equal to the degree of the corresponding vertex f_i^* in G^*.

6.Let n, m, f denote the number of vertices, edges and regions of a connected plane graph G and n^*, m^*, f^* are the same of the dual graph G^*.Then,
$n^* = f, m^* = m, f^* = n$.

18.3.2 Euler's Formula

Euler's formula ($n - e + f = 2$) is the basic counting tool relating vertices, edges, and faces in planar graphs.

Theorem 18.10. *If a connected plane graph G has exactly n vertices, e edges, and f faces, then $n - e + f = 2$.*

Proof. We use induction on n, number of vertices.

Basis step($n = 1$): G is a "bouquet" of loops, each a closed curve in the embedding. If $e = 0$, then, $f = 1$, and the formula holds. Each added loop passes through a face and cuts it into two faces (by the Jordan Curve Theorem). This augments the edge count and the face count each by 1. Thus the formula holds when $n = 1$ for any number of edges.

Induction step($n > 1$): Since G is connected, we can find an edge that is not a loop. When we contract such an edge, we obtain a plane graph G' with n' vertices, e' edges, and f' faces. The contraction does not change the number of faces (we merely shorten the boundaries), but it reduces number of edges and vertices by 1, so $n' = n - 1$, $e' = e - 1$ and $f' = f$. Applying the induction hypothesis yields

$$n - e + f = n' + 1 - (e' + 1) + f' = n' - e' + f' = 2.$$

Fig. 18.10

□

Remark 18.11. 1. By Euler's formula, all planar embeddings of a connected graph G have the same number of faces. Although the dual may depend on the embedding chosen for G, the number of vertices in the dual does not.

2. Euler's formula as stated fails for disconnected graphs. If a plane graph G has k components, then adding $k - 1$ edges to G yields a connected graph without changing the number of faces. Hence Euler's formula generalizes for plane graphs with k components as $n - e + f = k + 1$.

Euler's formula has many applications, particularly for simple plane graphs, where all faces have length at least 3.

Theorem 18.12. If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If G is also triangle-free, then $e(G) \leq 2n(G) - 4$.

Proof. Suppose, we consider the number of edges surrounding a face. If we sum over these numbers, we count each edge twice since it is used to surround two faces. Thus we get that, $\sum_{j=1}^{f} f_j = 2e$, where $\{f_j\}$ are the list of face lengths. Now, every face boundary in a simple graph contains at least three edges, implying that $f_j \geq 3$ for all j. Hence, we get $2e = \sum_{j=1}^{f} f_j \geq 3f$.
Also, from Euler's formula we have $n - e + f = 2 \implies f = 2 + e - n$. Putting it in the inequality, $2e \geq 3f = 3(2 + e - n) \implies -e \geq -3n + 6 \implies e \leq 3n - 6$.
When the graph is triangle free, each face length is at least 4. In that case, we have, $2e \geq 4f \implies 2e \geq 4(2 + e - n) \implies -2e \geq -4n + 8 \implies e \geq 2n - 4$. □

Remark 18.13. We have proved nonplanarity of K_5 and $K_{3,3}$ using Jordan Curve Theorem. The same can be proved from the last theorem as well. Let us recall the image of the graphs first.

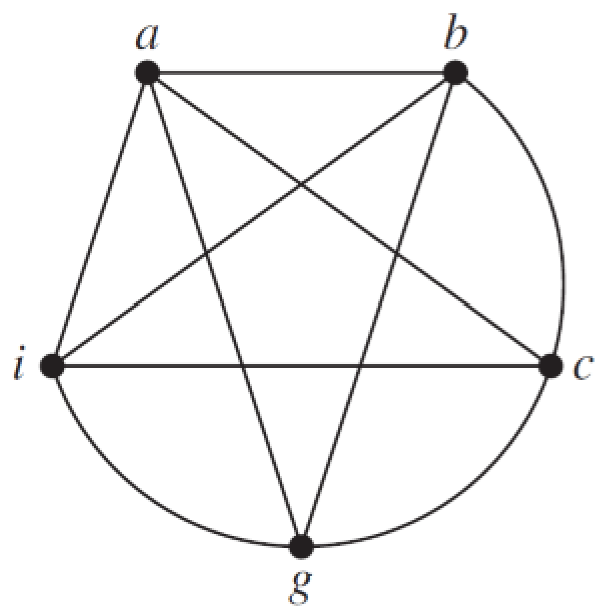

Fig. 18.11. K_5

Consider K_5. We have $n = 5, e = 10$. Therefore, $e = 10 > 9 = 3n - 6$. So, K_5 cannot be planar.

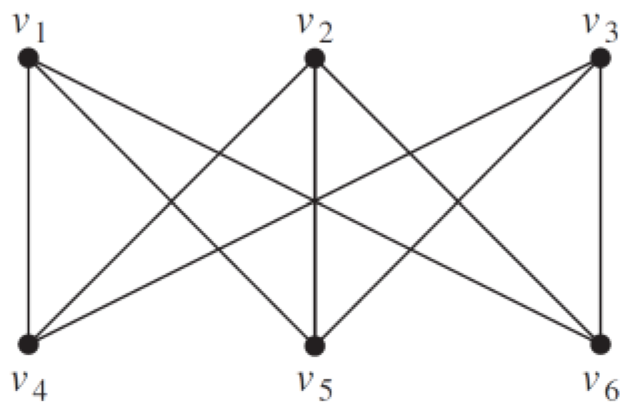

Fig. 18.12. $K_{3,3}$

Consider $K_{3,3}$. Note that it is triangle-free and here, $n = 6, e = 9$. Therefore, $e = 9 > 8 = 2n - 4$. So, $K_{3,3}$ cannot be planar too. These two graphs have too many edges to be planar.

18.4 Kuratowski's Theorem

We have seen that $K_{3,3}$ and K_5 are not planar. Clearly, a graph is not planar if it contains either of these two graphs as a subgraph. Surprisingly, all nonplanar graphs must contain a subgraph that can be obtained from $K_{3,3}$ or K_5 using certain permitted operations.

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an **elementary subdivision**. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions. Figure 16.7 gives some examples of homeomorphic graphs.

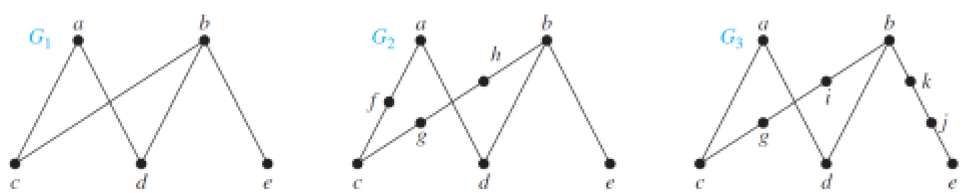

Fig. 18.13. Homeomorphic Graphs

Kazimierz Kuratowski established the following theorem which characterizes planar graphs using the concept of graph homeomorphism.

Theorem 18.14. A graph is nonplanar iff it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

The proof of the above theorem is too detail and complex to be included here. The proof is done in many standard textbooks and can be found going to this link.