Discrete Mathematics 01/04/2019

Lecture 16: Eulerian Tours and Hamiltonian Cycles

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1 Introduction

In this lecture, we will take up the problem of traversing through a graph through trails and cycles. Its a natural question to ask whether we can travel through a graph following its edges and visit certain(or all) vertices or edges. This question can serve as an abstraction to many other problems.
One might ask, whether it is possible to travel through a city visiting each town once, or maybe which states of a Markov chain are accessible from certain other states, or simply, which web pages maybe reached from other pages following the links contained in that page. Observe, all these problems boils down to the same abstract problem of traversing through the graph.
However, here we will consider only two specific instances of paths and trails, in this discussion.
Next we present a historical problem, that was one of the strongest motivation for Euler behind coming up with the theory of graphs.

1.1 The Könisberg bridge problem

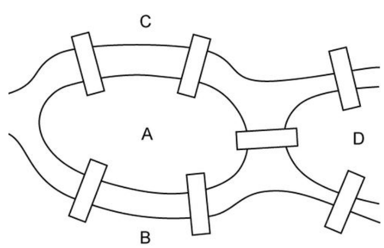

Figure 1: The city of Könisberg

The city of Könisberg in the Prussian empire had four blocks: A,B,C,D connected by seven bridges over the river Pregel. The map is shown in Figure 1. On Sundays, the citizens of Könisberg would promenade about the town, and the problem arose as to whether it was possible to plan a promenade in such a way that each bridge is crossed exactly once and only once.
This fairly non-intuitive problem is difficult to solve. This problem was a major motivation for Leonhard Euler behind his theory of graphs. Euler came up with a brilliant solution to this, which turns out to be one of the most important results in his first paper on graph theory. We will look at his clever way out to this problem, later in our discussion.
Euler came up with objects, that are called Eulerian trails in his honour, about which we

shall be studying now. The city of Könisberg now is a part of the Russian state by the name of Kaliningrad.

2 Tours and Eulerian trails

Definition 2.1. For a graph G, a closed walk covering all the edges is called a *tour*.

Definition 2.2. A walk covering each edge of a graph G exactly once is called an *Eulerian trail*. If an Eulerian trail is closed, its called an *Euler tour* or a *closed Eulerian trail* or an *Eulerian circuit* (since a closed trail is also called a circuit).

Let's consider some examples:

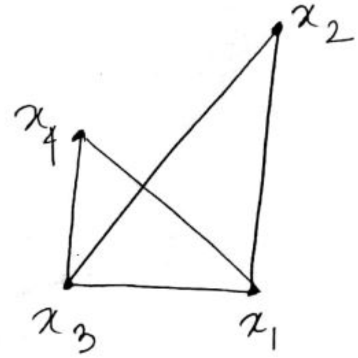

Figure 2

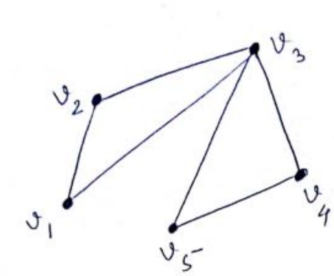

Figure 3

Example 2.3. In figure 2, observe that $x_1 - x_2 - x_3 - x_1 - x_4 - x_3 - x_1$ is a tour, since its closed and covers each edge once, however its not an Eulerian tour since it repeats the edge $\{x_1, x_3\}$.
$x_1 - x_2 - x_3 - x_1$ is not a tour since it doesn't visit the edge $\{x_1, x_4\}$ and $\{x_3, x_4\}$.

Example 2.4. In figure 3, $v_1 - v_2 - v_3 - v_4 - v_5 - v_3 - v_1 - v_2$ is an Eulerian trail. In fact, its an Euler tour.

Definition 2.5. A graph having an Euler tour is called an *Eulerian Graph*.

The next theorem provides an easy condition to check whether a graph is Eulerian or not.

Theorem 2.6. *A connected graph is Eulerian iff each vertex has even degree.*

Proof. **only if part:**
Consider an Eulerian graph G=(V,E). Now by definition ,$\exists$ an Eulerian tour, say $\mathcal{C}$, starting and ending at say, $u \in V$. Consider a vertex $v \in V$
Then we have the following cases:

Case 1: $u \neq v$
Since, v is a part of $\mathcal{C}$, and noting that $\mathcal{C}$ is a closed walk and every edge occurs exactly once, we conclude that v must be entered and exited by two different edges. Thus, we can group the edges meeting v as $\{(e_i, f_i)\}$, for $i = 1, 2, \ldots n$, for some number n, satisfying,$\forall$ i, $e_i \neq f_i$ and $\forall i, j$ $e_i = e_j$ iff $f_i = f_j$ and $i = j$, where e_i and f_j are edges. As an example,

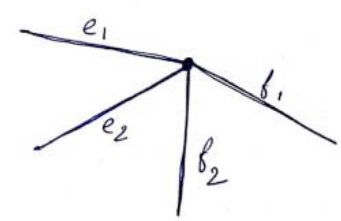

Figure 4

consider figure 4, showing edges meeting a certain vertex. We are grouping the edges as: $\{(e_1, f_1), (e_2, f_2)\}$. Note that then, d($v$)= $2n$, and hence is even.
Case 2: $u = v$
Note in this case, the start and the end edges contribute 2 to d(v). Except this, if v is repeated somewhere else in $\mathcal{C}$, then a similar argument as case 1 yields that the number of additional edges meeting v must be even. Hence,d(v) must be even.
if part:
Assume that the graph $G = (V, E)$, satisfies d(v) is even $\forall v \in V$.
Note that for any v, d(v)$\neq$ 0, else the graph would have been disconnected. So, the minimum degree of any vertex is at least 2. Thus $\delta(G) \geq 2$ and hence, $\exists$ a cycle $\mathcal{C}$ of length at least 3.
Let $n = |V|$. We will apply induction on n.
Base case: $n = 3$, and thus G is a triangle and hence is trivially Eulerian.
Induction hypothesis: Assume that $\forall n \leq m$, a graph with n many vertices with degree of each edge even, is Eulerian.
Induction step: Now consider $n = m + 1$.
Consider a graph G with $m + 1$ vertices, such that each vertex-degree is even. As noted earlier, G has a cycle $\mathcal{C}$.
Now, define $\overline{G} = G - \mathcal{C}$. Remove the isolated vertices of $\overline{G}$ to obtain graph H (Note that a vertex in $\overline{G}$ is isolated iff it was a part of cycle and had all its neighbour confined in the cycle $\mathcal{C}$).
Assume that H is composed of connected components $\{H_i\}$, for $i = 1, 2, \ldots k$ for some k. Since, G was connected, thus $\forall 1 \leq j \leq k$, $\exists$ vertex $v_{i_j} \in \mathcal{C}$ such that, $v_{i_j} \in H_j$. Observe that after removing the cycle $\mathcal{C}$, only 2 edges associated with the vertex v_{i_j} gets deleted. Thus its degree in H_j still remains even. Thus, the degree of each vertex in H_i is even $\forall i$. By induction hypothesis, each H_j has an Euler tour E_j, such that E_j starts and ends at v_{i_j}.
Construct an Euler tour for graph G as follows:
$\mathcal{E} = (v_1, e_2, \ldots, v_{i_1}, E_1, v_{i_1+1}, e_{i_1+2}, \ldots, v_{i_k}, E_k, v_{i_k+1}, e_{i_k+2}, \ldots, e_l, v_l)$
where, $\mathcal{C} = (v_1, e_2, v_3, \ldots, e_l, v_l)$ and obviously $v_l = v_1$.
What we are basically doing is starting from vertex v_1 of $\mathcal{C}$, and then going along the cycle, untill we encounter some common vertex with a connected component, say (WLOG) H_1.

We then complete the Euler tour E_1 and then continue along the cycle $\mathcal{C}$, untill we complete it. This closed walk we get is denoted $\mathcal{E}$ and by its construction, its immediate that its an Euler tour, and hence G is Eulerian.

Consider figure 5 for a simple demonstration of the proof. Here, C is the cycle, and H_1

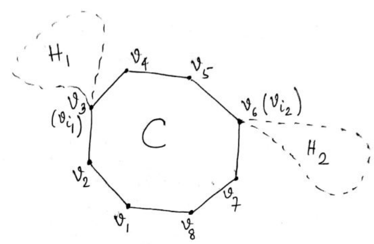

Figure 5: Demonstration of the proof of if part

and H_2 are the connected components left, after removing C and the isolated vertices. This completes the induction step, and hence proves the if part. □

Theorem 2.7. *A connected graph has an Eulerian trail iff there are atmost two odd degree vertices.*

Proof. **only if part:**
Consider a connected graph $G = (V, E)$ having an Euler tour H starting and ending at vertices u and v respectively. Consider any other vertex w other than u or v. Since H is an Euler tour, thus w must have occurred in H, somewhere other than the endpoints. By an argument similar to the only if part of the previous theorem, we conclude, w must have even degree. Thus, there can be atmost two vertices with odd degree, namely u and v. This proves the only if part.
if part:
Consider a connected graph G such that it has atmost two odd degree vertices.
Now in case, it has no odd degree vertices, then the previous theorem applies and hence we conclude the graph has an Eulerian trail (which is in fact a tour). Noting that a graph can have even many vertices of odd degree, we are only left with the case when G has exactly two odd degree vertices $\{u, v\}$.
Consider the graph $H = (V, E \cup \{u, v\})$, that is we are adding a new edge between u and v. Note, each vertex of H has an even degree and thus, the previous theorem implies there is a closed Eulerian trail $\mathcal{E}$ in H. If we remove the vertex $\{u, v\}$ from $\mathcal{E}$, we would get atleast an Eulerian trail $\mathcal{T}$ in G.
Hence under any case, G contains an Eulerian trail. This completes the proof. □

2.1 The Könisberg bridge problem: Solution

Here we present a simple solution to the Könisberg bridge problem (Euler), using the powerful tools we just developed.

Note that the map of the city of Könisberg can be reduced to figure 6. The problem asks whether this connected graph has an Eulerian trail or not. Now observe that the vertices D,

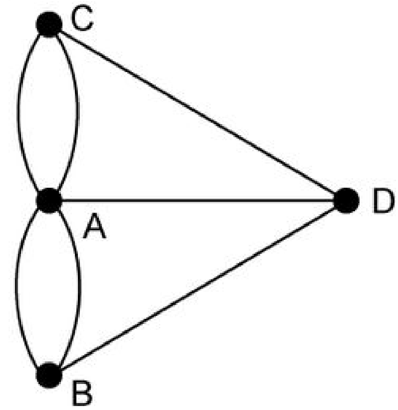

Figure 6: The city of Könisberg: Graph skeleton

C and B have degree 3 each, and A has degree 5. Since there are more than two (in fact all) vertices of odd degree in this connected graph, hence it cannot have an Eulerian trail. Thus, it was not possible for the people of Könisberg to make a tour of the entire city, with double-crossing a bridge!

3 Hamiltonian Path and Cycle

In the year 1859, the Irish mathematician Sir William R. Hamilton made a dodecahedron out of wood and assigned names of some important city to each of the vertices.The challenge was to find a route along the edges of dodecahedron that started from a city and ended on it, visiting every other city exactly once. This is famously known as the Hamilton's puzzle. The graph of a dodecahedron in given in figure 7.

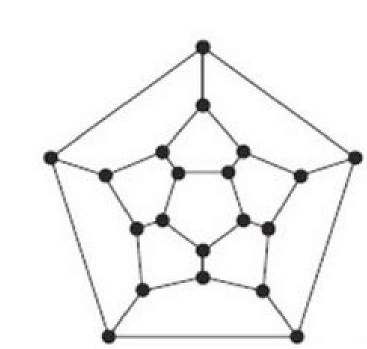

Figure 7: Graph of a dodecahedron

As we will see, this is about finding what is called a Hamiltonian cycle in the graph. Let's begin with some definitions.

Definition 3.1. *For a graph G, a path covering all of its vertices is called a Hamiltonian path.*

Definition 3.2. *A cycle covering all the vertices of a graph G, is called a Hamiltonian cycle.*

And quite obviously,

Definition 3.3. A graph is called *Hamiltonian*, if it contains a Hamiltonian cycle.

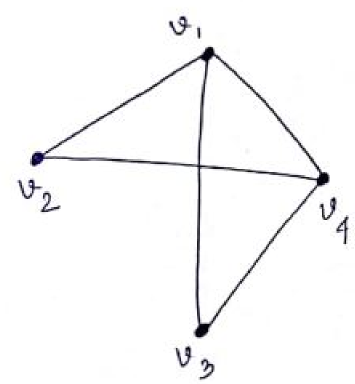

Figure 8

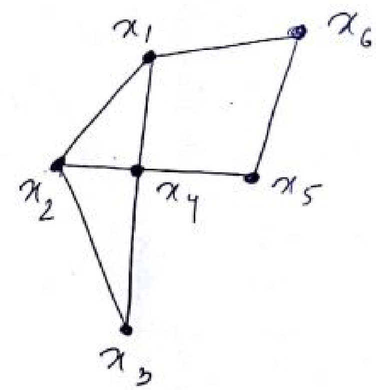

Figure 9

Example 3.4. *In figure 8, $v_2 - v_1 - v_4 - v_3$ is a Hamiltonian path.*
In figure 9, $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_1$ is a Hamiltonian cycle.

We might also be interested in graphs, that are not Hamiltonian but are closest to the Hamiltonian property. The next definition captures it:

Definition 3.5. A *Maximal non-Hamiltonian graph* is a non Hamiltonian graph G, such that the addition of an edge between any two non-adjacent edges, makes it Hamiltonian.

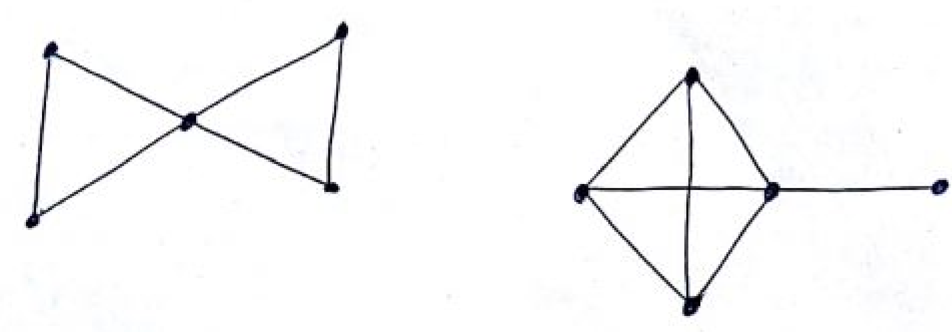

Figure 10

Example 3.6. *In figure 10, note that both the figures are maximally non-Hamiltonian. Addition of an edge between any two non-adjacent vertices makes it Hamiltonian, for both of them. The first one of them goes by the name butterfly graph and the next one by (4,1)-lollipop graph.*

Theorem 3.7. *Suppose $G = (V, E)$ is a simple graph with $n = |V| \geq 3$ vertices such that $\delta(G) \geq \frac{n}{2}$. Then G is Hamiltonian.*

Proof. Let's assume to the contrary that $\exists$ some non-Hamiltonian graph $G = (V, E)$ satisfying the conditions mentioned in the theorem. Note that we can add edges without violating the conditions on the lower bound of $\delta(G)$. Hence WLOG, assume that G is maximal non-Hamiltonian.
Let $V = \{v_1, v_2, \ldots v_n\}$.
Hence, $\exists$ atleast two non-adjacent vertices, u and v. Now form a new graph H by adding the edge $\{u, v\}$ to G. Then H has to be Hamiltonian and thus there is a Hamiltonian cycle $\mathcal{C}$. Remove the edge (if at all) from $\mathcal{C}$, to obtain $\mathcal{H}$, which is atleast a Hamiltonian path and is entirely contained in G. Thus G contains a Hamiltionian path of the form:
$\mathcal{H} : u = v_1 - v_2 - \cdots - v_n = v$.
Now define the following sets,
$A = \{v_i : u - v_{i+1}$ is an edge,
$B = \{v_j : v - v_j$ is an edge;
We make the following claims:
<u>**Claim 1:**</u> $v_n \notin A$
Pf: If this were the case, then we would have got a loop at $v_1 (= v_n)$, which contradicts the fact that the graph is simple. This proves the claim.
<u>**Claim 2:**</u> $v_n \notin B$
Pf: Similar to claim 1, if this were not the case, then we would have a loop at v_n. Thus claim 2 must be true.
<u>**Claim 3:**</u> $A \cap B = \phi$
Pf: If not, then $\exists$ k, such that $v_k \in A \cap B$. Then we have edges, $v_1 - v_{k+1}$ and $v_n - v_k$ respectively. Thus we have a Hamiltonian cycle
$\mathcal{C} : v_1 - v_{k+1} - v_{k+1} - \cdots - v_n - v_k - v_{k-1} - \cdots - v_1$
which is a clear contradiction to our assumption that G is non-Hamiltonian. This proves the claim.
And thus from claim 3, $|A \cup B| = (|A| + |B|) < n$ ($\because$ atleast v_n is left out from both A and B).
However, note that the lower bound on the degree posed by the condition in the theorem suggests that $|A| \geq \frac{n}{2}$ and $|B| \geq \frac{n}{2} \implies |A| + |B| \geq n$ which gives a contradiction.
This completes the proof □

Let's look at some corollaries.

Corollary 3.8. *Let G be a simple graph with n vertices, and two non-adjacent vertices u and v such that d(u)+d(v)≥ n. Then, G is Hamiltonian iff G' formed by adding the edge $\{u, v\}$ to G, is Hamiltonian.*

Proof. The only if part follows trivially. G is a Hamiltonian graph, and hence has a Hamiltonian cycle. With addition of an edge, the same cycle remains Hamiltonian in the new

graph G'. Hence, G' is a Hamiltonian graph.
Now consider the only if part.
Assume to the contrary that G' is Hamiltonian whereas G is not. Now since G' is Hamiltonian, hence there is a Hamiltonian cycle $\mathcal{C}$ in G'. Delete the edge $\{u, v\}$(if at all present) from $\mathcal{C}$, to obtain a Hamiltonian path $\mathcal{P}$.
Now, note that $\because \mathcal{P}$ doesn't contain the edge $\{u, v\}$, it must also be a Hamiltonian path for the graph G. Now, observe the edge $\{u, v\} \notin \mathcal{P}$, and by definition, $\mathcal{P}$ visits each vertex of G exactly once. Hence we may assume, $\mathcal{P}$ has the following form:
$\mathcal{P} : u = v_1 - v_2 - \cdots - v_{n-1} - v_n = v$, where $\{v_i : 1 \leq i \leq n\}$ is the set of vertices of G.
Claim: If for $i \in \{2, 3, \ldots, n-1\}$, u is adjacent to v_i, then v and v_{i-1} are not adjacent.
Pf: If this were the case, then the path: $v_1 - v_i - v_{i+1} - \cdots - v_n - v_{i-1} - v_{i-2} - \cdots - v_1$ is a Hamiltonian cycle in G, which is a contradiction. Hence, our claim must be true.
Thus by virtue of the claim, each of the vertices $v_i,\ i \in \{2, 3, \ldots, n-1\}$, must be adjacent to atmost one of u or v. And thus $d(u) + d(v) < n$ which is a contradiction to our assumption that the sum is atleast n.
This proves the if part. □

Next we consider what is called the closure of a graph.

Definition 3.9. Let G be a graph with n vertices, then the *closure* of G (denoted by $c(G)$) is the graph obtained by adding edges between non-adjacent vertices whose degree sum is atleast n, untill this can no longer be done.

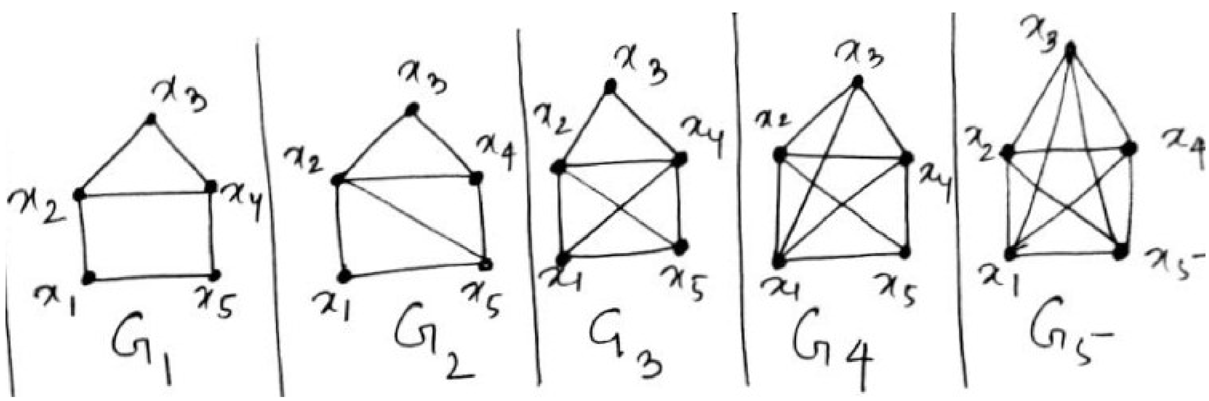

Figure 11

Example 3.10. *Consider figure 11. We are starting with a graph G_1 with 5 vertices. We will obtain its closure. Consider the non-adjacent vertices x_2 and x_5. Their degree sum is 5. Thus, we add an edge between them to obtain graph G_2. Now in G_2, the non-adjacent vertices x_1 and x_4 have degree sum 5, and hence we add an edge between them to obtain G_3. In this new graph x_3 and x_1 are non-adjacent with degree sum 5, hence we add an edge between them to obtain G_4. Similarly, we add an edge between x_3 and x_5 in G_4 to obtain G_5. No non-adjacent vertices are left in G_5. Hence, $c(G_1)$=G_5. In this regard, it should be pointed out that, since there are multiple ways to arrive at the closure of a graph, the* *problem of uniqueness must be settled before we make any further claim. This is done in the next lemma.*

Lemma 3.11. *For a graph G, $c(G)$ does not depend on the order in which we chose to add edges when more than one is available.*

Proof. Suppose G_1 and G_2 are obtained as $c(G)$ from G by two different implementations of the closure procedure. Let $n = |V[G]|$.
Let $e_1, e_2, \ldots, e_s$ and $f_1, f_2, \ldots, f_t$ denote the sequence of edges added to G to make G_1 and G_2, respectively.
Claim: Every edge of G_1 is in G_2 and vice-versa, i.e, $e_i \in E[G_2]$ and $f_j \in E[G_1]\ \forall\ i, j$.
Pf: Suppose not. Let $e_{k+1} = u - v$ be the first edge of G_1 not in G_2. Consider graph H obtained from G by adding edges $e_1, e_2, \ldots, e_k$. Then:

- $e_{k+1} \in E[G] \implies$ in H, $d(u) + d(v) \geq n$.
- H is a subgraph of G_2. So, $d(u)$ in $G_2 \geq d(u)$ in H, and $d(v)$ in $G_2 \geq d(v)$ in H.

It follows that $d(u) + d(v)$ in G_2 is $\geq d(u) + d(v)$ in $H \geq n$. Thus, E_{k+1} should be an edge of G_2, which is a contradiction.
This proves the claim.
$\therefore E[G_1] = E[G_2]$ and since, they contain the same set of vertices (i.e, the set of vertices of G), we must have $G_1 = G_2$, and thus $c(G)$ is well defined. □

Hence, the closure of a graph is unique. We now turn to our next corollary.

Corollary 3.12. *(Bondy-Chavátal) G is Hamiltonian iff $c(G)$ is Hamiltonian*

Proof. Note that by definition of closure, we see that $\exists$ graphs $G_0, G_1, \ldots, G_k$, such that $G_0 = G$ and $G_k = c(G)$ and G_i is obtained from G_{i-1}, by adding edges to the later, adhering to the conditions imposed by the definition of closure $\forall 1 \leq i \leq k$. Now, the previous corollary implies that G_{i-1} is Hamiltonian iff G_i is Hamiltonian. And thus induction implies $G = G_0$ is Hamiltonian iff $c(G) = G_k$ is Hamiltonian. □

This concludes our discussion on Hamiltonian Paths and Cycles.

4 Complexity Issues

We've uptill now discussed some theoretical results on occurrence of Eulerian trails and Hamiltonian cycles in graphs. Now let us turn to the practical problem of actually deciding whether a given graph G is Eulerian or Hamiltonian.
Lets turn to them one by one.

4.1 Deciding whether G is Eulerian

Theorem 2.6 provides us with an easy method for this problem when G is connected. We just need to check whether each vertex has even degree or not. If G has n vertices, then the most obvious way for checking this would be:

- *Consider a vertex v.*
- *Check every other vertices u for adjacency and find d(v). If d(v) is odd, the graph is not Eulerian. Else, proceed.*
- *Repeat untill all vertices have been checked.*

If all the vertices pass the test, the graph is Eulerian.
From the second step we observe, we are making $n-1$ *comparisons, for each vertex* v. *Hence a total of* $n(n-1)$ *comparisons. Thus the problem can be solved in* $O(n^2)$ *time.*

4.2 Deciding whether G is Hamiltonian

Deciding upon whether a graph is Hamiltonian, turns out to be a difficult problem (in fact, its NP complete). We didn't come across any necessary and sufficient condition for it.
Theorem 3.7 gives a sufficient condition for a graph to be Hamiltonian. By an argument similar to the previous section, we conclude, the conditions of this theorem can be checked in $O(n^2)$ *time as well.*
However, if this condition fails to hold, we have no confidence that the graph is not Hamiltonian.
In this context, it would be interesting to look upon another such graph theoretic NP hard problem, that goes by the following name:

4.3 The Travelling Salesman Problem (TSP)

Statement: *Given a list of cities, with a given cost of travelling between a pair of cities. If it is possible to travel from any city to any other city, which route will minimise the expense of a salesman, who wants to travel each city once, and return back to his original starting point?*
A better graph theoretic formulation of the above problem maybe:
"Given a complete graph K_n *with preassigned weights to each edge, which sequence of edges will travel each vertex atleast once, minimising the sum of the weights?"*

The above problem has been studied for decades. However, an efficient algorithm is still unknown.