**Discrete Mathematics** 20/03/2019

# Lecture 14: Degree and Degree Sequence

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## 1 Introduction

In this lecture, we will briefly discuss on neighbourhood and degree of a vertex in a graph. Then, there will come some results related to degree of vertices of a graph. Finally, we will define graphic sequence and prove a theorem on graphic sequence reduction.

## 2 Neighbourhood of a vertex

**Definition 2.1.** *(Neighbourhood of a vertex $v$)* For the graph $G = (V, E)$ *neighbourhood of the vertex* $v \in V$ (denoted by $N(v)$) is the set of all vertices which are adjacent to vertex $v$.That is;

$$N(v) = \{u \in V : \exists \text{ edge (u,v)} \in E\}$$

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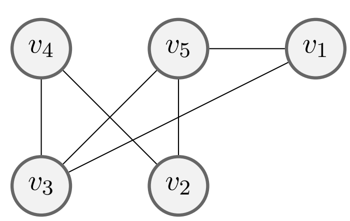


In this figure,
N($v_1$) = $\{v_3, v_5\}$;
N($v_2$) = $\{v_4, v_5\}$;
N($v_3$) = $\{v_1, v_4, v_5\}$;
N($v_4$) = $\{v_2, v_3\}$;
N($v_5$) = $\{v_1, v_2, v_3\}$.

## 3 Some terminologies related to degree

**Definition 3.1.** *(Degree of a vertex in an un-directed graph)* The *degree of a vertex in an un-directed graph* is the number of edges incident with it, except that a loop at the vertex contributes twice to the degree of it. Degree of the vertex v is denoted by $deg(v)$ or $d(v)$.

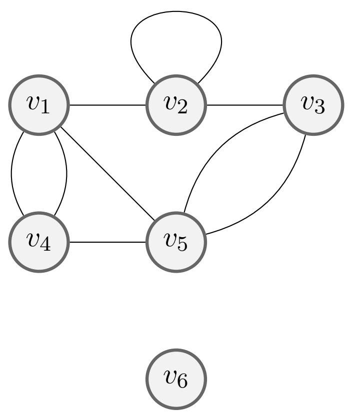
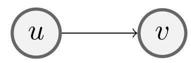



Figure 1: un-directed graph

In this figure,
d($v_1$) = 4;
d($v_2$) = 4;
d($v_3$) = 3;
d($v_4$) = 3;
d($v_5$) = 4;
d($v_6$) = 0.

**Definition 3.2.** *(Initial and terminal vertex of an edge in a digraph)* When $(u, v)$ is an edge of a directed graph or digraph $G$, $u$ is said to be adjacent to $v$ and $v$ is said to be adjacent from $u$.The vertex $u$ is called the *initial vertex* of the edge $(u, v)$, and $v$ is called the *terminal or end vertex* of it. The initial vertex and terminal vertex of a loop are the same.




**Definition 3.3.** *(In-degree and out-degree of a vertex in a digraph)* In a digraph the *in-degree* of a vertex $v$, denoted by $deg^{-}(v)$ or $d^{-}(v)$, is the number of edges with $v$ as their terminal vertex. The *out-degree* of $v$, denoted by $deg^{+}(v)$ or $d^{+}(v)$, is the number of edges with $v$ as their initial vertex. (Note that a loop at a vertex contributes 1 to each of the in-degree and the out-degree of the vertex.)

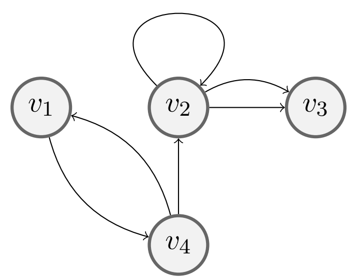



Figure 2: In-degree and out-degree of a vertex in a digraph

In this figure,
$deg^{-}(v_1) = 1$;
$deg^{+}(v_1) = 1$;
$deg^{-}(v_2) = 2$;
$deg^{+}(v_2) = 3$;
$deg^{-}(v_3) = 2$;
$deg^{+}(v_3) = 0$;
$deg^{-}(v_4) = 1$;
$deg^{+}(v_4) = 2$.

**Definition 3.4.** *(Degree of a vertex in a digraph)* In a digraph, *degree of a vertex* $v$, denoted by $deg(v)$ or $d(v)$ is the sum of the in-degree and out-degree of that vertex.That is;

$$deg(v) = deg^{-}(v) + deg^{+}(v)$$

In figure 2,
$\deg(v_1) = 2$;
$\deg(v_2) = 5$;
$\deg(v_3) = 2$;
$\deg(v_4) = 3$.

## 4 Some results related to degree

**Theorem 4.1.** *Given a graph* $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

**Proof:** *For un-directed graph,* each edge connecting two vertices $u$ & $v$ contributes 1 to the degree of $u$ & 1 to the degree of $v$. Also a loop contributes 2 degrees to a vertex.
*For digraph,* each edge with initial vertex $u$ & terminal vertex $v$ contributes 1 to the out-degree of $u$ and 1 to the in-degree of $v$. Also, a loop at a vertex contributes 1 to each in its in-degree and out-degree.
Hence, each edge of the graph contributes 2 degrees to the number of total degrees of all the vertices.
Hence,
2 * Number of edges = total number of degrees of the graph (both in directed and un-directed graph).

**Theorem 4.2.** *Number of odd degree vertices in a graph is even.*

**Proof:** From the previous theorem, we know that total number of degrees of all the vertices of a graph is even.
Let $U$ be the set of all odd degree vertices and $W$ be the set of all even degree vertices.
Therefore, $\sum\limits_{v \in U} d(v) + \sum\limits_{v \in W} d(v) =$ An even number.
Now, as each term in the sum $\sum\limits_{v \in W} d(v)$ is even, hence the total sum is also even.
Therefore, $\sum\limits_{v \in U} d(v)$ is even.
But each term of the summation $\sum\limits_{v \in U} d(v)$ is odd. So, in order to make the total sum even, there should be even number of these odd terms.
In other words, number of odd degree vertices in a graph is even.

**Theorem 4.3.** *For a simple graph with at least two vertices, there must be at least two vertices having the same degree.*

**Proof:** Let $G = (V, E)$ be a simple graph (that is, no self-loops and no parallel edges).
Suppose, $|V| = n$.
Therefore, for any $v \in V$, $d(v) \in \{0, 1, ..., (n-1)\}$.
But, if any of the $n$ vertices has degree 0, that means it is connected to no other vertex of the graph.
Therefore, no other vertex of the graph can have the degree $(n-1)$.
Similarly, if any vertex of the simple graph has degree $(n-1)$, it means it connects to all other edges of the graph and therefore, no vertex can have degree 0.
That means, the degrees of the $n$ vertices either take values from {0, 1,...,(n-2)} or from {1,...,(n-1)}.
In each case, there are $n$ vertices taking values from a set of $(n-1)$ elements.
therefore, by PHP, two of them will take the same value.
Hence, at least, two of the vertices of the simple graph have the same degree.

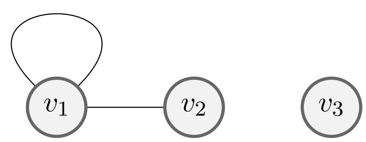
## 5 Degree sequence

**Definition 5.1.** *(Degree sequence)* A sequence ($d_1$, $d_2$,..., $d_n$) of non-negative integers is called a *degree sequence* if $d_1 \geq d_2 \geq ... \geq d_n$, with $\sum_{i=1}^{n} d_i$ = even and in the sequence ($d_1$,$d_2$,...,$d_n$) there are even many odd terms.

Note: Given a degree sequence of length $n$, we can always construct a graph with $n$ vertices taking degree from the degree sequence, but the graph is not necessarily a simple one.

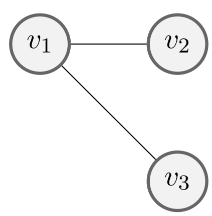
**Definition 5.2.** *(Graphic sequence)* A degree sequence of length $n$ is called a *graphic sequence* if there exists a simple graph with $n$ vertices taking degrees from the degree sequence.

Example 1: $(3, 1, 0)$ graphic?




But this is not a simple graph as $v_1$ has a loop. So, the degree sequence is not graphic.

Example 2: $(2, 1, 1)$ graphic?




This is indeed a simple graph. Therefore, $(2, 1, 1)$ is a graphic sequence.

## 6 An important theorem on graphic sequence reduction

**Theorem 6.1.** *A degree sequence ($d_1$, $d_2$,..., $d_n$) of $n$ elements ($n \geq 2$) where $d_1 \geq d_2 \geq ... \geq d_n$ is graphic if and only if ($d_2$ - 1, $d_3$ - 1,..., $d_k$ - 1, $d_{k+1}$ - 1, $d_{k+2}$ - 1,..., $d_n$) is a graphic sequence of $(n - 1)$ elements where $k = d_1$. [Note that for n = 1, the only one element graphic sequence is $d_1$ = 0 ].*

**Proof:** For $n = 1$, we already know that the one element degree sequence $\{d_1\}$ will be graphic iff $d_1 = 0$.

So, take $n \geq 2$.

*First we will prove the* ***if*** *part.*
Given, the degree sequence $d = (d_1, d_2,..., d_n)$ with $d_1 \geq d_2 \geq ... \geq d_n$, define $d' = (d_2$ - 1, $d_3$ - 1,..., $d_k$ - 1, $d_{k+1}$ - 1, $d_{k+2}$ - 1,..., $d_n)$ .
Now, suppose $d'$ is graphic. That is, $\exists$ a simple graph $G'$ with $(n-1)$ vertices with degree sequence $d'$ with the elements of $d'$ arranged in descending order.
Then, in $G'$, add one extra vertex adjacent to the first $k$ many vertices having degrees $d_2$ - 1, $d_3$ - 1,..., $d_k$ - 1, $d_{k+1}$ - 1 respectively.
The new graph $G$ still remains simple with $n$ many vertices taking their degrees from the degree sequence $d$.
Hence, $d$ is also a graphic sequence.
*This completes the* ***if*** *part.*

*Now, we will prove the* ***only if*** *part.*
Given a simple graph $G$ with $n$ many vertices and degree sequence $d$, we have to construct a simple graph with $(n-1)$ many vertices and degree sequence $d'$.
Let $v$ be the vertex with the highest degree $d_1$ in $G$.
Let $S$ be the set of the $k$ vertices in $G$ having degrees $d_2$, $d_3$,..., $d_{k+1}$.
Now, there will be two cases.
**Case 1:** Neighbourhood of the vertex $v$, $N(v) = S$.
In that case, just delete the vertex $v$ with all its $k$ connections to each of the elements of $S$ to get the desired simple graph $G'$ with $(n-1)$ vertices and degree sequence $d'$.
**Case 2:** Some vertex of $S$ is missing from $N(v)$.
In that case, we modify $G$ to increase $|N(v) \cap S|$ without changing any vertex degree.
Suppose, $u \in S$ is missing from $N(v)$. But in order to keep the $deg(v) = k$, there must be another vertex $w$ of $G$, $w \notin S$ such that $v$ connects to $w$.
Claim: $\exists$ at least one vertex $x \in N(u)$ such that $w$ and $x$ are not connected.
proof of the claim: If for all $x \in N(u)$, $w$ and $x$ were connected, then, $deg(w) \geq deg(u) + 1$ [this extra one degree comes from the edge between $w$ and $v$ when there is no edge between $u$ and $v$].
But, $S$ contains the $k$ highest degrees of $G$ below $d_1$ and $u \in S \implies deg(w) \leq$ deg(u) [contradiction].
Hence the claim is true.
Therefore, among the four vertices $v, u, w, x$; $v$ and $w$ are connected and $u$ and $x$ are connected.
Now, get a new graph by switching the edges between the four points; i.e., delete the edges between $(v, w)$ and $(u, x)$ and connect $(v, u)$ and $(w, x)$ instead.
In this way, the vertex degrees remain the same but now $u \in N(v) \cap S$.
Since modifying in this way, $|N(v) \cap S|$ can increase at most $k$ times, repeating this process converts $G$ into another graph $G_1$ that has degree sequence same as $d$ but $N(v) = S$.
Therefore, we just return to *case 1* and get the desired simple graph $G'$ with $(n-1)$ vertices and degree sequence $d'$.
Hence, $d'$ is a graphic sequence.

*This completes the **only if** part.*

**Application of the previous theorem:** *Show that* $(5, 4, 4, 3, 2, 2)$ *is a graphic sequence.*

reduction 1: $(3, 3, 2, 1, 1)$.

reduction 2: $(2, 1, 0, 1)$.

(Now, we can just avoid the vertex with degree 0. That is, if $(2, 1, 1)$ is graphic, then $(2, 1, 0, 1)$ is also graphic. Because, in the simple graph having degree sequence $(2, 1, 1)$ just add another vertex with degree 0 to have a new simple graph with degree sequence $(2, 1, 0, 1)$.)

Hence, the actual reduction may be taken as $(2, 1, 1)$.

reduction 3: $(0, 0)$. Now, this is indeed a graphic sequence with a simple graph of two vertices each having degree 0.

Hence, $(5, 4, 4, 3, 2, 2)$ is a graphic sequence.