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Lecture 13: Ramsey Numbers

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13.1 Introduction

Ramsey's theorem is a fundamental result in combinatorics. The first version of this result was proved by F. P. Ramsey. This initiated the combinatorial theory now called Ramsey theory, that seeks regularity amid disorder: general conditions for the existence of substructures with regular properties. In this application, it is a question of the existence of monochromatic subsets, that is, subsets of connected edges of just one colour.

To begin with, let's discuss the following problem.

Proposition 13.1.1. (Friends and Strangers Theorem) *In any party of six people, either at least three of them are (pairwise) mutual strangers or at least three of them are (pairwise) mutual acquaintances.*

Proof : Let $G = (V, E)$ be a graph and $|V| = 6$. Fix a vertex $v \in V$. We consider two cases.

- If the degree of v is at least 3, then consider three neighbors of v, call them x, y, z. If any two among $\{x, y, z\}$ are friends, we are done because they form a triangle together with v. If not, no two of $\{x, y, z\}$ are friends and we are done as well.

- If the degree of v is at most 2, then there are at least three other vertices which are not neighbors of v, call them x, y, z. In this case, the argument is complementary to the previous one. Either $\{x, y, z\}$ are mutual friends, in which case we are done. Or there are two among $\{x, y, z\}$ who are not friends, for example x and y, and then no two of $\{v, x, y\}$ are friends.

We will see later that 6 is the minimum number to enjoy the above property. If $|V| < 6$ then it is not always true that there is either a group of 3 mutual friends or there is a group of 3 mutual strangers.

More generally, we consider the following setting. We color the edges of K_n (a complete graph on n vertices) with a certain number of colors and we ask whether there is a complete sub-graph (a clique) of a certain size such that all its edges have the same color. We shall see that this is always true for a sufficiently large n. Note that the question about friendships corresponds to a coloring of K_6 with 2 colors, *friendly* and *unfriendly*. Equivalently, we start with an arbitrary graph and we want to find either a clique or the complement of a clique, which is called an independent set. This leads to the definition of Ramsey numbers.

13.2 Definitions of Ramsey Property

Definition 13.2.1. *A positive integer N is said to have (p, q)-Ramsey Property, (where $p, q \in \mathbb{N}$) if the following holds,*

Version 1: *Let $G = (V, E)$ with $|V| = N$ be a graph.For any partition $\{X, Y\}$ of the set of two element subsets of V, there exists,*

Either a p-element subset S_1 of V such that all 2-element subsets of $S_1 \in X$

or, a q-element subset S_2 of V such that all 2-element subsets of $S_2 \in Y$.

Version 2: *Given any graph $G = (V, E)$ with $| V | = N$,*

Either G will have a clique of size p (K_p) or $\overline{G}$ will have a clique of size q (K_q).

Version 3: *In any edge-coloring of K_N with 2 colors, namely, color 1 and color 2, there will be either a K_p with all edges having color 1 or K_q with all edges with color 2.*

Definition 13.2.2. ***(Ramsey Number)*** *The minimum of all integers that satisfy (p, q)-Ramsey Property is called the (p, q)-Ramsey Number or $R(p, q)$.*

Note that it is not clear that Ramsey numbers are finite! Indeed, it could be the case that there is no finite number satisfying the conditions of $R(p, q)$ for some choice of p,q. However, the following theorem proves that this is not the case and gives an explicit bound on $R(p, q)$.

13.3 Theorem on Finiteness of Ramsey Number

Before going to the theorem, let us observe some basic properties:

1. For every $p, q \in \mathbb{N}$ we have $R(p, q) = R(p, q)$

   Proof. Observe that,it suffices to show the following:
   n satisfies (p, q)-Ramsey property, if and only if n satisfies (q, p)-Ramsey property.
   Fix $p, q \in \mathbb{N}$
   Let, n be a number such that it satisfies (p, q)-Ramsey property. Consider any edge of K_n with two colors (red and blue). Then by the definition of Ramsey Property, either it contains K_p with all the edge colors red or it contains K_q with all the edge colors blue.

   Now, if we invert the coloring, then we will have either it contains K_q with all the edge colors red or it contains K_p with all the edge colors blue.
   Since, any edge coloring can be obtained by inverting the colors of some other coloring, We conclude, any coloring of K_n has a red K_q or blue K_p.

   Hence, n satisfies (q, p)- Ramsey property.

   □

2. For every $n \geq 1$ we have $R(n, 1) = R(1, n) = 1$

   Proof. Observe that, one simple graph is possible with $|V| = 1$. [Just one point and $E = \phi$].So,it always contains an independent set of size 1.

   Hence,1 satisfies (n,1)-Ramsey property and 1 is the least positive integer. So by the first property we can conclude $R(n, 1) = R(1, n) = 1$. □

3. For every $n \geq 2$ we have $R(n, 2) = R(2, n) = n$

   Proof. Fix $n \geq 2$.First, we will show that, n satisfies (n,2)-Ramsey property. Consider a graph $G = (V, E)$ with $|V| = n$, Then there are two possibilities:

   - G is complete graph (or $G = K_n$).Then G contains a clique of size n.
   - $G \neq K_n$.Then there exist two disjoint vertices $(v_1, v_2) \in V$ such that $(v_1, v_2) \notin E$.Then $\{v_1, v_2\}$ will be an independent set of size 2.

   Therefore, n satisfies (n,2)-Ramsey property.

   Now, we will show that (n-1) does not satisfy (n,2)-Ramsey property.It completes the proof.

   Consider, $G = K_{n-1}$ ($|V| = n - 1$). Clearly G does not contain any n-clique.As all the vertices are connected to each other G does not contain any independent set of size 2.

   Hence, $(n - 1)$ does not satisfy $(n, 2)$-Ramsey property. So, n is the minimum number which satisfy $(n, 2)$-Ramsey property and therefore $R(n, 2) = n$. By the first property we can conclude $R(n, 2) = R(2, n) = n$. □

Theorem 13.3.1. *For any $p, q \geq 1$, there is $R(p, q) < \infty$ such that any graph on $R(p, q)$ vertices contains either a clique of size p or an independent set of size q.*

Proof. Fix $p, q \in \mathbb{N}$ and let, $p + q = m$
Our induction hypothesis is that, $R(s, t) < \infty \ \forall s, t \in \mathbb{N}$ such that $s + t < m$.
We show that, $R(p, q) \leq R(p-1, q) + R(p, q-1)$. To see this, let $n = R(p-1, q) + R(p, q-1)$ and consider any graph G on n vertices. Fix a vertex $v \in V$. We consider two cases:

- There are at least $R(p, q-1)$ edges incident with v. Then we apply induction on the neighbors of v, which implies that either they contain an independent set of size p, or a clique of size $q-1$. In the second case, we can extend the clique by adding v, and hence G contains either an independent set of size p or a clique of size q.
- There are at least $R(p-1, q)$ non-neighbors of v. Then we apply induction to the non-neighbors of v and we get either an independent set of size $(p-1)$, or a clique of size q. Again, the independent set can be extended by adding v and hence we are done.

Since $p + (q-1) < m$ and $(p-1) + q < m$, $R(p, q-1) < \infty$ and $R(p-1, q) < \infty$, and it follows by strong induction that $R(p, q)$ is finite. □

Corollary: Here we introduce a stronger bound than Ramsey's original bound. First, we claim that, $R(p, q) \leq \binom{p+q-2}{p-1}$. Clearly, it holds for the base cases where $p = 1$ or $q = 1$ since every graphs contains a clique or an independent set of size 1. The inductive step is as follows

$R(p, q) \leq R(p-1, q) + R(p, q-1) \leq \binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}$
by Pascal's rule.

Problem 13.3.1. *Prove* $R(3,3) = 6$

Answer. *We have almost done this problem twice but without the concept of Ramsey number.It follows from proposition 13.2.1 (and also the explanation after solution) of this scribe or the last problem of previous scribe (Introduction to Graph Theory). So, either there exist a friendship-coloured (say red) triangle or a stranger-coloured triangle (say blue) in* K_6 *.Equivalently, either there exist a clique of size 3 with red or a clique of size 3 with blue. So, 6 satisfies (3,3)-Ramsey property.*
Claim: *If N satisfies* (p,q)*-Ramsey property then N+1 satisfies* (p,q)*-Ramsey property.*

Proof. Edges of a complete graph K_N is colored by red and blue color.There exist either a clique of size p (K_p) with all edges colored red or a clique of size q (K_q) with all edges colored blue.Without loss of generality assume there exist a red colored K_p. If we introduce a new vertex (make it K_{N+1}) whatever color (red or blue) we use to color new N edges of K_{N+1} the K_p remains red. □

As we know the minimum of all integers that satisfy (p,q)*-Ramsey Property is called the* $R(p,q)$ *from the above claim if we can disprove* $R(3,3) = 5$ *then the proof will be done immediately.*

counterexample: *Take 5 vertices of* K_5 *of a pentagon then connect all the sides of pentagon by red and connect all the diagonals by blue.There is no same colored triangle containing vertices. Hence* $R(3,3) = 6$*.*

13.4 Generalized Ramsey Number

Definition 13.4.1. *A positive integer N is said to have $(p_1, p_2, \cdots, p_k)$-Ramsey Property, (where $p_1, p_2, \cdots, p_k \in \mathbb{N}$ and with the colours $c_1, c_2, \cdots, c_k$) if given any coloring of the complete graph K_N,*
Either it will have a clique of size p_1 (K_{p_1}) with all edges colored c_1
or it will have a clique of size p_2 (K_{p_2}) with all edges colored c_2
or it will have a clique of size p_3 (K_{p_3}) with all edges colored c_3

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or it will have a clique of size p_k (K_{p_k}) with all edges colored c_k.
The minimum of all integers that satisfy $(p_1, p_2, \cdots, p_k)$-Ramsey Property is called the $(p_1, p_2, \cdots, p_k)$ Ramsey Number or $R(p_1, p_2, \cdots, p_k)$.

Theorem 13.4.2. *For any $p_1, ..., p_c \geq 1$, there is $R(p_1, ..., p_c) < \infty$ such that for any c-coloring of the edges of $K_n, n \geq R(p_1, ..., p_c)$, there is a clique of size p_i in some color i.*

Proof. We will show that:

$$R(p_1, \cdots, p_c) \leq R(p_1, \cdots, p_{c-2}, R(p_{c-1}, p_c))$$

We note that the RHS only contains only Ramsey numbers for $c - 1$ colors and 2 colors, and therefore exists. Thus it is the finite number t, by the inductive hypothesis. So proving this will prove the theorem.

Consider a graph on t vertices and color its edges with c colors. Now "go color-blind" and pretend that $c - 1$ and c are the same color.

Thus the graph is now $(c - 1)$-colored.

By the inductive hypothesis, it contains either:

- a complete monochromatic graph K_{p_i} with color i for some $1 \leq i \leq (c - 2)$,
  or
- a complete monochromatic graph $K_R(p_{c-1}, p_c)$-colored in the "blurred color".

In the former case we are finished.

In the latter case, we recover our sight again and see from the definition of $R(p_{c-1}, p_c)$ we must have either:

- a complete $(c-1)$-monochromatic graph $K_{p_{c-1}}$,
  or

- a complete c-monochromatic graph K_{p_c}.

In either case the proof is complete.

□

Problem 13.4.1. *Each of 17 students talked with every other student. They all talked about three different topics. Each pair of students talked about one topic. Prove that there are three students that talked about the same topic among themselves.*

Answer. *The problem can be simplified as "Prove $R(3,3,3) = 17$" and We use $R(3,3) = 6$ (which we have proved essentially in Friends and Strangers Theorem) here.*

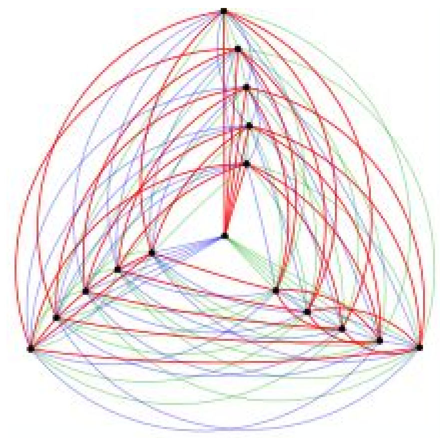
Denote the topics discussed by the students T_1, T_2, T_3. Consider one of the students, say A. Since A talked about three topics to 16 other students, by the Pigeonhole Principle, there are at least 6 students with whom A talked of a single topic, say, T_1. Then there are just to possibilities.

- *If any pair from the 6 students discussed topic T_1 we are done.*

- *Otherwise, there are 6 students that discussed between themselves only 2 topics - T_2 or T_3. So we are looking at the number R(3,3) which is 6; and we are done in this case also.*

As we know if we prove that N satisfies $(p_1, ..., p_c)$-Ramsey property $N+1, N+2, \cdots$ also satisfy $(p_1, ..., p_c)$-Ramsey Property. So If we can prove 16 doesn't satisfy $(3,3,3)$-Ramsey Property it implies $R(3,3,3) = 17$.

Claim: *16 does not satisfy* $(3,3,3)$*-Ramsey Property*
counterexample:
We have exhibit a coloring of K_{16} *in three colors with no monochromatic triangle.*



So the claim has been proved and we conclude $R(3,3,3) = 17$.

13.5 Schur's Theorem

Ramsey theory for integers is about finding monochromatic subsets with a certain arithmetic structure. It starts with the following theorem of Schur (1916), which turns out to be an easy application of Ramseys theorem for graphs.

Theorem 13.5.1. *For any* $k \geq 2$ *, there is* $n > 3$ *such that for any k-coloring of* $\{1, 2, \cdots, n\}$ *,there are three integers* x, y, z *of the same color such that* $x + y = z$*.*

Proof. We choose $n = R_k(3, 3, \cdots, 3)$, i.e. the Ramsey number such that any k-coloring of K_n. contains a monochromatic triangle.
Given a coloring c : $[n] \to [k]$, we define an edge coloring of
K_n: the color of edge $\{i, j\}$ will be $\chi(i, j) = c(|j - i|)$.
By the Ramsey theorem for graphs, there is a monochromatic triangle $\{i, j, k\}$; assume $i < j < k$. Then we set
$x = j - i$, $y = k - j$, $z = k - i$
We have $c(x) = c(y) = c(z)$ and so $x + y = z$. □

Schur used this in his work related to Fermat's Last Theorem. More specifically, he proved that Fermat's Last Theorem is false in the finite field $\mathbb{Z}_p$ for any sufficiently large prime p.

Theorem 13.5.2. *For every $m \geq 1$, there is p_0 such that for any prime $p \geq p_0$, in $\mathbb{Z}_p$ the congruence*

$$x^m + y^m = z^m$$

has a solution.

Proof. The multiplicative group $\mathbb{Z}_p^*$ is known to be cyclic and hence it has a generator g. Each element of $\mathbb{Z}_p^*$ can be written as $x = g^{mj+i}$ where $0 \leq i < m$. We color the elements of $\mathbb{Z}_p^*$ by m colors, where $c(x) = i$ if $x = g^{mj+i}$. By Schur's theorem, for p sufficiently large, there are elements $x, y, z \in \mathbb{Z}_p^*$ such that $x' + y' = z'$

$c(x') = c(y') = c(z')$
Therefore, $x' = g^{mj_x+i}$, $y' = g^{mj_y+i}$, $z' = g^{mj_z+i}$ and
$g^{mj_x+i} + g^{mj_y+i} = g^{mj_z+i}$
Setting $x = g_x^j$, $y = g_y^j$, $z = g_z^j$, we get a solution of $x^m + y^m = z^m$ in $\mathbb{Z}_p$.

□