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Lecture 11: Chains and Antichains

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11.1 Introduction

We recall that a *partial order* is a binary relation $\leq$ over a set X satisfying

- *(Reflexivity)* For all $x \in X$, $x \leq x$.
- *(Anti-Symmetry)* For $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
- *(Transitivity)* For $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

The set X endowed with the partial order $\leq$ is called a *partially ordered set* or *poset*. We often write $(X, \leq)$ to represent it.

To fix notation, let us write $x < y$ for two elements $x, y \in X$ satisfying $x \leq y$ and $x \neq y$.

We also recall that $x \in X$ is a *maximal element* if there does not exist $y \in X$ with $x < y$. Similarly $x \in X$ is a *minimal element* if there does not exist $y \in X$ with $y < x$. For the poset $(X, \leq)$, we denote the set of maximal elements by $Max(X)$ and the set of minimal elements by $Min(X)$.

The aim of the current lecture is to understand a result due to Dilworth that generalises the famous Erdös-Szekeres theorem in combinatorics. We need a few definitions, which we present as follows. From now on, we will always work in the poset $(X, \leq)$.

For $x, y \in X$, we say that x and y are *comparable* if either $x \leq y$ or $y \leq x$.

Definition 11.1. (Chain) A subset $C \subset X$ is called a **chain** if and only if x, y are comparable for all $x, y \in C$. The poset $(C, \leq)$ is thus a **totally ordered set** (TOSET).

Thus if the chain C has k elements then they can be written as $c_1 < c_2 < \cdots < c_k$.

Definition 11.2. (Antichain) A subset $A \subset X$ is called an **antichain** if and only if no different $x, y \in A$ are comparable.

Example: Let $X = \{(0,0), (0,1), (1,0), (1,1)\}$. Define the partial order $\preceq$ on X by $(x_1, x_2) \preceq (x_3, x_4)$ if and only if, both $x_1 \leq x_3$ and $x_2 \leq x_4$ ($\leq$ is the usual partial order on $\mathbb{R}$). Then $C = \{(0,0), (1,1)\}$ is a chain in $(X, \preceq)$, and $A = \{(0,1), (1,0)\}$ is an antichain in $(X, \preceq)$.

We record a few straightforward lemmas.

Lemma 11.3. *Any subset of a chain is a chain and any subset of an antichain is an antichain.*

Proof. If C is a chain in $(X, \leq)$ and $D \subset C$ then any two elements in D are comparable because they are comparable in C. Thus any subset of a chain is a chain.

Let A is an antichain in $(X, \leq)$ and $B \subset A$. If two elements in B are comparable, then they would be comparable in A as well, contradicting the fact that A is an antichain. $\square$

Lemma 11.4. *For any antichain A and any chain C, $|A \cap C| \leq 1$.*

Proof. Suppose not. Then there exist $x \neq y \in A \cap C$. Since $x, y \in A$, they are not comparable, but as $x, y \in C$, they are comparable, a contradiction. Thus, $|A \cap C| \leq 1$. $\square$

Lemma 11.5. *$Max(X)$ and $Min(X)$ are antichains.*

Proof. We present the proof only for $Max(X)$ since the proof for $Min(X)$ is similar. Let $x \neq y \in Max(X)$. Then as $x \in Max(X)$, we cannot have $x < y$ (and hence we cannot have $x \leq y$), and similarly, as $y \in Max(X)$, we cannot have $y < x$ (and hence we cannot have $y \leq x$). But this means that x and y are not comparable. This shows that $Max(X)$ is an antichain. $\square$

11.2 Main Theorems

We now present two major theorems. Without mentioning every time, we will assume we are working in the poset $(X, \leq)$.

Theorem 11.6. ***(Mirsky's Theorem)*** *Let r be the size of a largest chain in X. Then X can be partitioned into r and no fewer antichains.*

Proof. Let C be a largest chain in X, then $|C| = r$. Suppose X can be partitioned into p many antichains $A_1, \cdots, A_p$. Then using Lemma 11.4,

$$r = |C| = \sum_{i=1}^{p} |C \cap A_i| \leq p$$

This shows X can't be partitioned into fewer than r antichains.

We next show that we can get exactly r many antichains partitioning X. Define the following subsets of X:

$$\begin{aligned} X_1 = X, &\ \ A_1 = Min(X_1) \\ X_2 = X_1 \setminus A_1, &\ \ A_2 = Min(X_2) \\ X_3 = X_2 \setminus A_2, &\ \ A_3 = Min(X_3) \\ \vdots \quad \vdots \quad \vdots &\quad \vdots \quad \vdots \quad \vdots \end{aligned}$$

and we stop at $p \in \mathbb{N}$ such that $X_p \neq \phi$ but $X_{p+1} = \phi$.

For each i, $A_i = Min(X_i) \subset X_i$ is an antichain by Lemma 11.5. Observe that by the way the A_i have been constructed, $A_i \cap A_j = \phi$ for all $i \neq j$. Further, since $X_{p+1} = \phi$, it follows that $A_p = X_p$. Now

$$X = X_1 = A_1 \cup X_2 = A_1 \cup A_2 \cup X_3 = \cdots = \bigcup_{i=1}^{p-1} A_i \cup X_p = \bigcup_{i=1}^{p} A_i.$$

Therefore, $\{A_1, \cdots, A_p\}$ is a partition of X. We will show that $p = r$.

We have already seen that $p \geq r$. Suppose $p > r$. For any $2 \leq i \leq p$, if $x \in A_i$,then $x \in X_i = X_{i-1} \setminus A_{i-1}$ so $x \in X_{i-1}$ but $x \notin A_{i-1}$. This means there exists $y \in X_{i-1}$ such that $y < x$. If $y \notin A_{i-1}$ then $y \in X_i$. This would imply $x, y \in X_i$, $y < x$, contradicting $x \in A_i$. Thus $y \in A_{i-1}$.

This shows that for each $2 \leq i \leq p$, we can choose $a_i \in A_i$ such that $a_{i-1} < a_i$. This gives us a chain $a_1 < a_2 < \cdots < a_p$ in X, with length $p > r$, contradicting the assumption that any largest chain in X has length r. This shows $p = r$ necessarily. The proof is complete. □

Next we present the dual version of Theorem 11.6. It is due to Dilworth.

Theorem 11.7. ***(Dilworth's Theorem)*** *Let m be the size of the largest antichain in X. Then X can be partitioned into m and no fewer chains.*

Proof. That X cannot be partitioned into fewer than m chains can be proved in the same way as in Theorem 11.6. So we only show here that we can get exactly m many chains partitioning X.

We proceed by induction on the size of the set X. If $|X| = 1$, then the largest antichain is X itself, and the only possible chain is X again, so the result is true.

Now let $|X| \geq 2$. Assume the theorem to be true for all sets of size smaller than $|X|$. Let A be a largest antichain in X.

We consider two cases.

Case 1: Suppose every largest antichain in X is either $Max(X)$ or $Min(X)$. Hence, in particular, A is either $Max(X)$ or $Min(X)$. We choose $a \leq b \in X$ (equality allowed) such that $a \in Min(X)$ and $b \in Max(X)$ as follows. For any $a \in Min(X)$, we consider the set $S_a := \{y \in X | a \leq y\}$. Clearly $S_a \neq \phi$ as $a \in S_a$. S_a is a poset, so choose $b \in Max(S_a)$. Note, if $S_a = \{a\}$ then $b = a$ is the only choice. Then $b \in Max(X)$ because if there is $c \in X$ with $b < c$, then $a < c$, showing $c \in S_a$, contradicting $b \in Max(S_a)$. Notice that $\{a, b\}$ is a chain in X.

Let $Y = X \setminus \{a, b\}$. If Y has an antichain A' of size m, then A' must be an antichain (in fact a largest antichain) in X of size m, and $A' \neq Max(X)$ or $Min(X)$ since $a, b \notin A'$. This contradicts the assumption that every largest antichain in X is either $Max(X)$ or $Min(X)$.

Thus there cannot be any antichain in Y of size m. Of course there cannot be any antichain in Y of size greater than m, since such an antichain would be an antichain in X and we assumed m is the size of any largest antichain in X. So a largest antichain in Y must have size at most $m-1$. Now one of the sets $Max(X)\setminus\{b\}$ or $Min(X)\setminus\{a\}$ has size $m-1$ (by our assumption) and both these two sets are antichains in Y, hence Y has a largest antichain of size $m-1$. By construction, $|Y| < |X|$. So, by induction hypothesis, there exist $m-1$ disjoint chains $C_1, \cdots, C_{m-1}$ partitioning Y. Then the representation

$$X = \bigcup_{i=1}^{m-1} C_i \bigcup \{a, b\}$$

is a partition of X into m chains. This completes Case 1.

<u>Case 2:</u> Suppose A is not $Max(X)$ or $Min(X)$. Then we define

$$A^+ = \{y \in X : a \leq y \text{ for some } a \in A\}$$

and

$$A^- = \{y \in X : y \leq a \text{ for some } a \in A\}.$$

Since $A \neq Max(X)$, $A^- \neq X$ and hence $|A^-| < |X|$. Since $A \neq Min(X)$, $A^+ \neq X$ and hence $|A^+| < |X|$.

If $x \in X$ is incomparable with all elements in A, then $A \cup \{x\}$ is a larger antichain than A, which is a contradiction. This shows that $A^+ \cup A^- = X$.

If there exists $x \in X$ such that $x \leq a$ for some $a \in A$ and $b \leq x$ for some $b \in A$, then $b \leq a$, a contradiction as A is an antichain. Therefore, $A^+ \cap A^- = A$.

Let $A = \{a_1, \cdots, a_m\}$. Now A^+ is a poset such that $A = Min(A^+)$ and A is a largest antichain in A^+. (This is because any antichain in A^+ larger than A would be an antichain in X larger than A.) Thus A^+ can be partitioned into chains $S_1, \cdots, S_m$ where S_i is a chain originating from a_i. Similarly, A^- is a poset such that $A = Max(A^-)$ and A is a largest antichain in A^-. So, A^- can be partitioned into chains $T_1, \cdots, T_m$ where T_i originates from a_i.

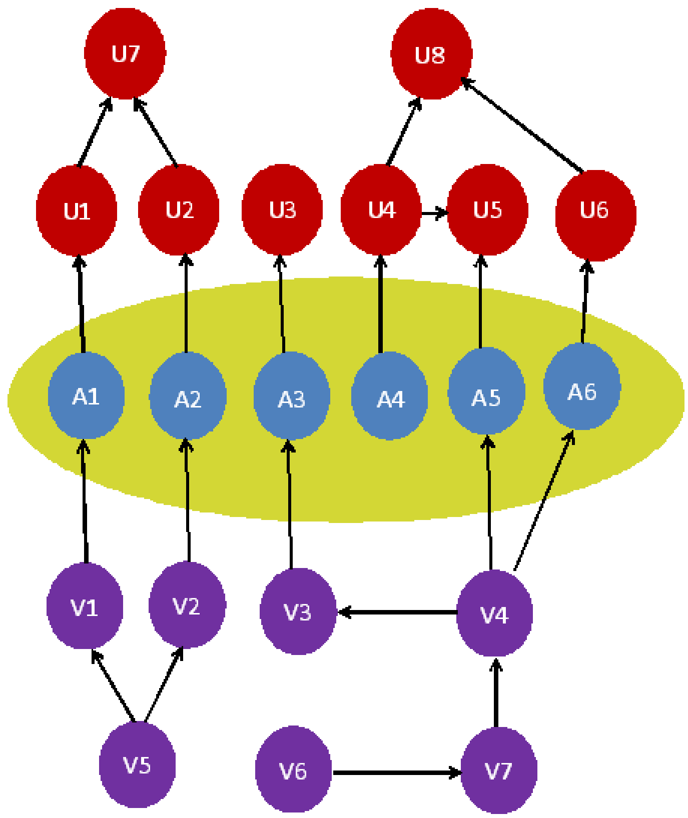
Next we note that $C_i := S_i \cup T_i$ is a chain. Suppose $x, y \in S_i \cup T_i$. If $x, y \in S_i$ or T_i it is clear that x and y are comparable. If $x \in S_i, y \in T_i$ then $y \leq a_i$ and $a_i \leq x$ implying $y \leq x$. Similarly x and y are comparable if $x \in T_i$ and $y \in S_i$.

Further we observe that for $i \neq j$, if $x \in S_i \cap T_j$ then $x \leq a_j$ and $a_i \leq x$ implying $a_i \leq a_j$ contradicting that A is an antichain. Thus $S_i \cap T_j = \phi$ whenever $i \neq j$. Similarly $T_i \cap S_j = \phi$. We already know that $S_i \cap S_j = \phi$ and $T_i \cap T_j = \phi$ for $i \neq j$. This shows $C_i \cap C_j = \phi$.

Also, $\bigcup_{i=1}^m C_i = \bigcup_{i=1}^m S_i \cup \bigcup_{i=1}^m T_i = A^+ \cup A^- = X$.

Thus, X can be partitioned into the chains $C_1, \cdots, C_m$.

The proof is now complete. □



The diagram above represents a poset where two points x and y are joined by a straight line directed from x to y if and only if $x \leq y$. The set of blue points (the set is coloured in green) is an example of an antichain A. Thus $A = \{A1, A2, \cdots, A6\}$. A^+ is the set of red and blue points, and A^- is the set of purple and blue points. Thus, $A^+ = \{A1, A2, \cdots, A6, U1, U2, \cdots, U8\}$ and $A^- = \{A1, A2, \cdots, A6, V1, V2, \cdots, V7\}$.

In what follows, we denote a chain in A^+ starting from a point $x \in A$ as $C(x)$ and a chain in A^- ending at a point $y \in A$ as $T(y)$. Thus,

$C(A1) = \{A1, U1, U7\}, T(A1) = \{V5, V1, A1\}$
$C(A2) = \{A2, U2\}, T(A2) = \{V2, A2\}$
$C(A3) = \{A3, U3\}, T(A3) = \{V6, V7, V4, V3, A3\}$
$C(A4) = \{A4, U4, U8\}, T(A4) = \{A4\}$
$C(A5) = \{A5, U5\}, T(A5) = \{A5\}$
$C(A6) = \{A6, U6\}, T(A6) = \{A6\}$

11.3 Application

We now present an application of the above theorem.

Theorem 11.8. ***(Generalized Erdös-Szekeres Theorem)*** *For any sequence of at least distinct $rs+1$ real numbers, there is either an increasing subsequence of length at least $r+1$ or a decreasing subsequence of length at least $s+1$.*

Proof. Let $x_1 < \cdots < x_n$ be a sequence of real numbers where $n \geq rs+1$. On the set $X = \{x_1, \cdots, x_n\}$ define the following partial order: $x_i \preceq x_j$ if and only if $i \leq j$ and $x_i \leq x_j$.

We first understand the chains and antichains in this poset. We claim, the chains are precisely the increasing subsequences, and the antichains are precisely the decreasing subsequences.

Suppose $x_{i_1} < \cdots < x_{i_k}$ be an increasing subsequence so that $i_1 < \cdots < i_k$. Therefore, $x_{i_1} \preceq \cdots \preceq x_{i_k}$ and hence this increasing subsequence is a chain. Now suppose C is a chain and let $C = \{x_{i_1}, \cdots, x_{i_k}\}$ where $i_1 < \cdots < i_k$. Then as any two elements are comparable, for $r < s$, either $x_{i_r} \preceq x_{i_s}$ or $x_{i_s} \preceq x_{i_r}$. Since $i_r < i_s$, we must have $x_{i_r} < x_{i_s}$. Thus whenever $r < s$ we have $x_{i_r} < x_{i_s}$ implying C is an increasing subsequence.

Suppose $x_{i_1} > \cdots > x_{i_k}$ is a decreasing subsequence so that $i_1 < \cdots < i_k$. So for any $r < s$ we have $i_r < i_s$ but $x_{i_r} > x_{i_s}$ so that x_{i_r} and x_{i_s} are not comparable. Thus any decreasing subsequence is an antichain. Let A be a chain and let $A = \{x_{i_1}, \cdots, x_{i_k}\}$ where $i_1 < \cdots < i_k$. No two elements are comparable. So for any $r < s$, since we already have $i_r < i_s$ we are forced to have $x_{i_r} > x_{i_s}$. Thus A is a decreasing subsequence.

Now assume there does not exist any increasing subsequence of length $\geq r+1$. This means, there does not exist any chain of length $\geq r+1$ so any largest chain has length at most r. Thus there exists a partition of X into at most r many antichains, that is, there can be at most r disjoint decreasing subsequences partitioning X. By pigeonhole principle, since $|X| \geq rs+1$, there must be at least $s+1$ elements in some antichain, which in turn implies there must be a decreasing subsequence of length at least $s+1$. The proof is complete. □

Special case: Now we will discuss the case when $r = s = n$.Then we can say that given any sequence of n^2+1 distinct integers there exist a monotone subsequence of $n+1$ integers. This is the famous Erdös-Szekeres Theorem.