Discrete Mathematics 27/02/2019

Solution to Mid-Semester Examination

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1. (a) (2 points) Is the *complement* of a set unique? Justify.

   (b) (8 points) Count the numbers of *equivalence* relations and *partial order* relations on the set $\mathbb{A} = \{1, 2, 3\}$

   (c) (4 points) Does $\mathbb{R}$ and $\mathbb{C}$ have the same *cardinality*? Prove your claim.

Solution:

(a) Complement of a set is not unique as the universal set is not defined. So if $\mathbb{N}$ is the set concerned, then its complement is $\mathbb{R} - \mathbb{N}$ if $\mathbb{R}$ is the universal set, whereas, $\mathbb{Z} - \mathbb{N}$ is the complement when $\mathbb{Z}$ is the universal set.

(b) The number of equivalence relation on set with cardinality n is the number of ways we can partition n. Here, $A = \{1, 2, 3\}$. The number of ways we can partition 3 is $3, 2 + 1, 1 + 1 + 1$. We can partition to 3 in 1 way, in $2 + 1$ in 3 ways and $1 + 1 + 1$ in 1 way. Hence the total number of equivalence relation is 5.
For the number of partial orders, we just list all the possible cases
1. no pair related: just one way to do that,

2. $1 < 2$, with 3 incomparable: 3 ways to choose the small element, 2 ways to choose the large, 6 orders all told,

3. a linear order, like $1 < 2 < 3$: 6 orders like this,

4. one element larger than the other two, those two being incomparable: 3 orders like this, from the 3 ways to choose the big element, and

5. one element smaller than the other two, those two being incomparable: 3 orders like this, from the 3 ways to choose the small element.

All told, that's $1 + 6 + 6 + 3 + 3 = 19$.

(c) Consider $f : \mathbb{C} \to \mathbb{R}^2$, such that $f(a + ib) = (a, b),\ \forall\ (a + ib) \in \mathbb{C}$. Clearly this is a bijective function. Hence, we can conclude that $\mathbb{C}$ and $\mathbb{R}^2$ are equivalent, i.e. $\mathbb{C} \sim \mathbb{R}^2$. Let, $g : (0, 1) \to \mathbb{R}$, be a function such that $g(x) = \tan(\pi x - \frac{\pi}{2})$. $g(x)$ takes all values of $\mathbb{R}$, and hence is an onto function. $\pi x - \frac{\pi}{2}$ takes values $(-\frac{\pi}{2}, \frac{\pi}{2})$ in the interval $(0, 1)$. As $\tan x$ is one one function in $(-\frac{\pi}{2}, \frac{\pi}{2})$ we have $g(x)$ to be a bijective function. Thus $\mathbb{R} \sim (0, 1)$. Hence, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \sim (0, 1) \times (0, 1) = (0, 1)^2$. Consider, $p : (0, 1) \times (0, 1) \to (0, 1)$ such that
$g(0.x_1x_2x_3 \cdots, 0.y_1y_2y_3 \cdots) = 0.x_1y_1x_2y_2x_3y_3 \cdots$. Note that p is an injective function. Hence, $(0, 1)$ dominates $(0, 1) \times (0, 1)$. Again consider $q : (0, 1) \to$

$(0,1) \times (0,1)$ such that $q(x) = (x, \frac{1}{2})$. This is also an injection and hence $(0,1) \times (0,1)$ dominates $(0,1)$. So, $(0,1)^2 \sim (0,1)$. Hence,

$$\mathbb{C} \sim \mathbb{R}^2 \sim (0,1)^2 \sim (0,1) \sim \mathbb{R}$$

So, $\mathbb{R}$ and $\mathbb{C}$ have same cardinality.

2. (a) (2 points) Is the *least* element in a POSET necessarily unique? Justify.

   (b) (3 points) Can there exist a POSET with multiple *minimal* elements, but only one *least* element? Justify.

   (c) (3 points) Find the fallacy in the following application of *strong induction*. Claim: Given $a \in \mathbb{R}^+$, one has that $a^n = 1$, $\forall n \in \mathbb{N}$(assume that $\mathbb{N}$ includes 0). In the proof, show that base case for $n = 0$. Assume that $\forall k \leq n$, it holds. And now show that $a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$.

   (d) (2 points) If a *logical theory* is *inconsistent*, what can we say about its *completeness*?

Solution:

(a) Suppose the least element is not unique. Say, x and y are two least elements in the POSET , say A. According to definition of least element , $x \leq a \; \forall a \in A$ and $y \leq a \; \forall a \in A \implies x \leq y$ and $y \leq x \implies x = y$. Thus we conclude that the least element is unique.

(b) Let the POSET be A. A minimal element say m in a POSET is such that $\nexists a \in A$ such that $a < m$. Now let us assume there are multiple minimal elements, say $m_1, m_2, \cdots m_p$ and a least element x. Now from the definition of least element,we get, $x \leq m_i \; \forall i = 1(1)p$ but from the definition of minimal elements we have $x \nless m_i \; \forall i = 1(1)p \implies x = m_i \forall i = 1(1)p$, which is a contradiction as m_i's are unique. Hence, there cannot be a POSET with multiple minimum elements but one least element.

(c) Given $a \in \mathbb{R}^+$ one has to show that $a^n = 1 \; \forall n \in \mathbb{N}$(assuming that $\mathbb{N}$ includes 0). To use strong induction, we start with base condition. When $n = 0$, $a^0 = 1$ which is true. But for $n = 1$, we get $a^1 = a \neq 1 \forall a \in \mathbb{N} - \{1\}$. Thus we cannot use strong induction in this problem.

(d) A logical theory is inconsistent when any statement in the theory and its negation can be proved using the axioms of the theory. A logical theory is complete when every true statements in the theory can be proved using the axioms. Thus completeness contains inconsistency. Thus if a logical theory is inconsistent then it is also complete.

3. (a) (4 points) Count the number of *arrangements* of n distinct letters in n distinct envelopes so that exactly one letter goes to the correct envelope.

(b) (12 points) Count the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 18,$$

that satisfy

$$1 \le x_1 \le 5, \quad -2 \le x_2 \le 4, \quad 0 \le x_3 \le 5, \quad 3 \le x_4 \le 9$$

using two methods : *inclusion-exclusion* principle and the method of *generating functions.*

Solution:

(a) Suppose that there are n letters numbered $1, 2, ..., n$. Let there be n envelopes also numbered $1, 2, ..., n$. We have to find the number of ways in which no letter goes to the envelope having same number as its number.Such an arrangement is known as a *derangement*. Suppose we want to count D_n, the number of derangements of $\{1, \cdots, n\}$.

Let S be the set of all permutations of $\{1, \cdots, n\}$, and

let T_i be the set of permutations which leave letter i in its natural position.

Then $D_n = |T_1^c \cap \cdots \cap T_n^c|$

$= |S| - \sum_i |T_i| + \sum_{i<j} |T_i \cap T_j| - \sum_{i<j<k} |T_i \cap T_j \cap T_k| + \cdots + (-1)^n |T_1 \cap \cdots \cap T_n|$

$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n}(n-n)!$

$= n! - \dfrac{n!}{1!} + \dfrac{n!}{2!} - \dfrac{n!}{3!} + \cdots + (-1)^n \dfrac{n!}{n!} = n! \left[1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \cdots + (-1)^n \dfrac{1}{n!}\right].$

Now in this problem, we can choose 1 letter to go to the correct envelope in n ways. Then we use derangement on the remaining $n-1$ letters. Hence the solution is :

$$n \cdot D_{n-1} = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1} \frac{1}{n-1!}\right]$$

(b)

$$x_1 + x_2 + x_3 + x_4 = 18$$

$$1 \le x \le 5, \quad -2 \le x_2 \le 4, \quad 0 \le x_3 \le 5, \quad 3 \le x_4 \le 9$$

We take $y_1 = x_1 - 1$; $y_2 = x_2 + 2$ $y_3 = x_3$; $y_4 = x_4 - 3$. Thus,

$$0 \le y_1 \le 4, \quad 0 \le y_2 \le 6, \quad 0 \le y_3 \le 5, \quad 0 \le y_4 \le 6$$

$$y_1 + y_2 + y_3 + y_4 = 16$$

Using **Inclusion Exclusion Principle**:

$A : (y_1, y_2, y_3, y_4)$ satisfying $y_1, y_2, y_3, y_4 \geq 0$
$P_1 : (y_1, y_2, y_3, y_4)$ satisfying and $y_1 \geq 5$
$P_2 : (y_1, y_2, y_3, y_4)$ satisfying and $y_2 \geq 7$
$P_3 : (y_1, y_2, y_3, y_4)$ satisfying and $y_3 \geq 6$
$P_4 : (y_1, y_2, y_3, y_4)$ satisfying and $y_4 \geq 7$
We want, $|P_1^c \cap P_2^c \cap P_3^c \cap P_4^c|$

$$= |A| - \sum_i |P_i| + \sum_{i<j} |P_i \cap P_j| - \sum_{i<j<k} |P_i \cap P_j \cap P_k| + |P_1 \cap P_2 \cap P_3 \cap P_4|$$

$$= \binom{19}{3} - \binom{14}{3} - \binom{13}{3} - 2\binom{12}{3} + \binom{8}{3} + 2\binom{7}{3} + 2\binom{6}{3} + \binom{5}{3} = 55$$

Using **Generating Functions**:

To find the coefficient of x^{16} in

$$(1 + x + \cdot + x^4)(1 + x + \cdot + x^5)(1 + x + \cdot + x^6)^2$$

$$= \frac{(1-x^5)(1-x^6)(1-x^7)^2}{(1-x)^4} = (1 - x^5 - x^6 + x^{11})(1 - 2x^7 + x^{14})(1-x)^{-4}$$

$$= (1 - x^5 - x^6 - 2x^7 + x^{11} + 2x^{12} + 2x^{13} + x^{14} + \cdots)(1-x)^{-4}$$

$$= \binom{19}{3} - \binom{14}{3} - \binom{13}{3} - 2\binom{12}{3} + \binom{8}{3} + 2\binom{7}{3} + 2\binom{6}{3} + \binom{5}{3} = 55$$

4. Use *generating functions* to
   (a) (4 points) count the number of n-bit sequences where both zeroes and ones appear even number of times.
   (b) (6 points) evaluate $2 + 8 + 24 + 64 + 160 + 384 + \cdots$ upto n terms.

Solution:

(a) Let $f(n)$ denote the number of n-bit sequences where both zeroes and ones appear even number of times. Clearly note that if n is odd, then there cannot be such a sequence as either number of zeroes or ones must be odd. Thus, $f(2n+1) = 0 \ \forall n \in \mathbb{N}$. Now, the generating function for number of 0 should be

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

The above generating function is like that because the number of 0 must be even and for k zeroes all the $k!$ permutations are identical, so we need to divide by $k!$. Similarly, the generating function for 1 is same. Thus $f(n)$ will be the coefficient of $\frac{x^n}{n!}$ in

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2$$

$$=\left(\frac{e^x+e^{-x}}{2}\right)^2=\left(\frac{2+e^{2x}+e^{-2x}}{4}\right)=\frac{1}{2}\left(1+\frac{e^{2x}+e^{-2x}}{2}\right)$$

The coefficient of $\frac{x^n}{n!}$ is $\frac{1}{2}\cdot 2^n=2^{n-1}$.

(b) Let a_k be the k-th term of the above sum and S_n be the n-th partial sum. With some observation we can get that $a_k=k2^k$. Let $F(x)$ be the generating function of the terms, i.e.

$$F(x)=\sum_{k\geq 0}a_kx^k=\sum_{k\geq 0}k(2x)^k$$

$$=2x+2.(2x)^2+3.(2x)^3+\cdots$$

$$=2x(1+2.(2x)+3.(2x)^2+\cdots)=2x(1-2x)^{-2}$$

Now let $G(x)$ be the generating function of the partial sums S_n i.e,

$$G(x)=\sum_{n\geq 0}S_nx^n$$

$$=\sum_{n\geq 0}(a_0+a_1+\cdots+a_n)x^n=F(x)(1+x+x^2+\cdots)$$

$$=F(x)(1-x)^{-1}=\frac{2x}{(1-2x)^2(1-x)}$$

Breaking the LHS into partial fractions we get ,

$$\frac{2x}{(1-2x)^2(1-x)}=\frac{2}{(1-2x)^2}-\frac{4}{(1-2x)}+\frac{2}{1-x}$$

$$=\sum_{n\geq 0}(n+1)2^{n+1}x^n-\sum_{n\geq 0}2^{n+2}x^n+\sum_{n\geq 0}2x^n$$

$$\implies \sum_{n\geq 0}S_nx^n=\sum_{n\geq 0}[(n-1)2^{n+1}+2]x^n$$

Thus, $S_n=(n-1)2^{n+1}+2$.

5. (a) (8 points) Solve the following *recurrence* relation:

$$a_n = 6a_{n-1} - 9a_{n-2} + (n^2+1)3^n, \quad \forall n \geq 2,$$

where $a_0 = 0, a_1 = 1$.

(b) (6 points) A *divide and conquer* algorithm works on an integer array of size n. For $n \geq 2$, it divides the array into two almost equal halves and recursively processes each part. After the *recursive calls* return, it takes constant time 1 (i.e, just one elementary operation) to combine the solutions on the parts. Formulate a recurrence for the time complexity function $t(n)$ and use *induction* on n to show that $t(n) \in \mathbb{O}(n)$. Note that n is any positive integer $n \geq 2$ and not necessarily a power of 2.

Solution:

(a) Let us first solve the homogeneous equation $a_n = 6a_{n-1} - 9a_{n-2}$ which gives,

$$x^2 = 6x - 9 \implies (x-3)^2 = 0$$

Thus the homogeneous solution is $a_n = c_1 3^n + c_2 n 3^n$. Now for the particular solution we take $a_n = n^2(an^2 + bn + c)3^n$ as the root 3 has multiplicity 2. Putting the value in the given recurrence relation, we get

$$n^2(an^2+bn+c)3^n$$

$$= 6(n-1)^2(a(n-1)^2+b(n-1)+c)3^{n-1} - 9(n-2)^2(a(n-2)^2+b(n-2)+c)3^{n-2} + (n^2+1)3^n$$

Simplyfying the above equation we get

$$(1-12a)n^2 + (24a-6b)n + (-14a+6b-2c+1) = 0$$

Comparing the coefficients we get, $a = \frac{1}{12}$; $b = \frac{1}{3}$; $c = \frac{11}{12}$. Hence, the general solution is

$$a_n = c_1 3^n + c_2 n 3^n + n^2\left(\frac{n^2}{12} + \frac{n}{3} + \frac{11}{12}\right)3^n$$

Given,$a_0 = 0$ and $a_1 = 1$, we get $c_1 = 0$ and $c_2 = -1$.Hence,

$$a_n = n^2\left(\frac{n^2}{12} + \frac{n}{3} + \frac{11}{12}\right)3^n - n3^n.$$

(b) $t(n)$ denotes the time complexity function of array size n. It takes 1 constant time to combine the solutions of the parts. The algorithm divides the array into almost two equal halves and recursively progresses. clearly $t(2) = 1$. Thus the recurrence relation is,

$$t(n) = t\left(\lfloor\frac{n}{2}\rfloor\right) + t\left(\lceil\frac{n}{2}\rceil\right) + 1$$

(the two near equal halves and 1 time for combining them)
When , $n = 2^k$, we have $t(2^k) = 2t(2^{k-1}) + 1$. If $t(n) \in \mathcal{O}(n)$ then

$$\limsup_{n\to\infty} \frac{t(n)}{n} < \infty$$

Since, $t(2) = 1$, $t(2) \in \mathcal{O}(n)$ .Thus the base case is true. So if $t(2^k) \in \mathcal{O}(n)$, then

$$t(2^{k+1}) = 2t(2^k) + 1$$
$$\implies \frac{t(2^{k+1})}{2^{k+1}} = \frac{t(2^k)}{2^k} + \frac{1}{2^k}$$
$$\implies \limsup_{k\to\infty} \frac{t(2^{k+1})}{2^{k+1}} = \limsup_{k\to infty} \left[\frac{t(2^k)}{2^k} + \frac{1}{2^k}\right]$$
$$\implies \limsup_{k\to\infty} \frac{t(2^{k+1})}{2^{k+1}} < \infty \text{ (since } t(2^k) \in \mathcal{O}(n))$$
$$\implies t(2^{k+1}) \in \mathcal{O}(n)$$

Now, if $2^k < n < 2^{k+1}$, then $t(2^k) < t(n) < t(2^{k+1})$. By Sandwich Theorem, we get $t(n) \in \mathcal{O}(n)$. Thus, $t(n) \in \mathcal{O}(n) \forall n$.

6. (a) (4 points) Let $T(r, n)$ be the number of onto *functions* from a set with cardinality r to a set with cardinality n. Prove the following recurrence using *combinatorial argument*:

$$T(r, n) = nT(r-1, n-1) + nT(r-1, n).$$

Algebraic derivation using any explicit formula would lead to zero credit.

(b) (4 points) Prove that the number of *p-partitions* of a positive integer n is equal to the number of *partitions* of $n + \binom{p}{2}$ into p *distinct* parts.

Solution:

(a) $T(r, n)$ is the number of onto functions from a set with cardinality r to a set with cardinality n. Choose an element from the set with cardinality r say a. Now, if a goes to an element say b then either (1) no other elements goes to b, or (2) atleast one other element goes to b. For (1) the total number of onto functions are $T(r-1, n-1)$ and for (2) the total number of onto functions are $T(r-1, n)$. Now we can choose b in n ways, hence we get the following recursion,

$$T(r, n) = n \cdot (T(r-1, n-1) + T(r-1, n))$$

(b) Let $\lambda_1, \lambda_2, \cdots \lambda_p$ be a partition of n such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. Then, $\lambda_1 + 1 > \lambda_2$; $\lambda_2 + 1 > \lambda_3$; $\cdots$; $\lambda_{p-1} + 1 \ > \lambda_p$.
Therefore,

$$\lambda_1 + (p-1) > \lambda_2 + (p-2) > \cdots > \lambda_{p-1} + 1 > \lambda_p$$

$$= \beta_1 > \beta_2 > \cdots > \beta_p.$$

We have, $\sum_{i=1}^{p} \beta_i = n + 1 + 2 + \cdots + (p-1) = n + \binom{p}{2}$. Hence, $\beta_1, \beta_2, \cdots \beta_p$ is a partition of $n + \binom{p}{2}$ into p distinct parts. Proved.