INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination
B. Stat II year, 2nd Sem, AY 2018–2019
Discrete Mathematics

Date: 20. 02. 2019, Time: 3 Hours (2:30 PM to 5:30 PM)

Total Marks: 72, Buffer Marks: 12, Maximum Marks: 60

**Please try to write all the part answers of a question at the same place.**

1. (a) Is the *complement* of a set unique? Justify.
   (b) Count the numbers of *equivalence* relations and *partial order* relations on the set $A = \{1, 2, 3\}$.
   (c) Do $\mathbb{R}$ and $\mathbb{C}$ have the same *cardinality*? Prove your claim.

   [2 + (4 + 4) + 4 = 14]

2. (a) Is the *least* element in a POSET necessarily unique? Justify.
   (b) Can there exist a POSET with multiple *minimal* elements, but only one *least* element? Justify.
   (c) Find the fallacy in the following application of *strong induction*. Claim: Given $a \in \mathbb{R}^+$, one has that $a^n = 1$, $\forall n \in \mathbb{N}$ (assume that $\mathbb{N}$ includes 0). In the proof, show the base case for $n = 0$. Assume that $\forall k \leq n$, it holds. And now show that $a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$.
   (d) If a *logical theory* is *inconsistent*, what can we say about its *completeness*?

   [2 + 3 + 3 + 2 = 10]

3. (a) Count the number of *arrangements* of $n$ distinct letters in $n$ distinct envelopes so that exactly one letter goes to the correct envelope.
   (b) Count the number of integral solutions of the equation

   $$x_1 + x_2 + x_3 + x_4 = 18,$$

   that satisfy

   $$1 \leq x_1 \leq 5, \quad -2 \leq x_2 \leq 4, \quad 0 \leq x_3 \leq 5, \quad 3 \leq x_4 \leq 9$$

   using two methods: *inclusion-exclusion* principle and the method of *generating functions*.

   [4 + (8 + 4) = 16]

4. Use *generating functions* to
   (a) count the number of $n$-bit sequences where both zeros and ones appear even number of times.
   (b) evaluate 2 + 8 + 24 + 64 + 160 + 384 + ... up to $n$ terms.

[4 + 6 = 10]

5. (a) Solve the following *recurrence* relation:

$$a_n = 6a_{n-1} - 9a_{n-2} + (n^2 + 1)3^n, \qquad \forall n \geq 2,$$

where $a_0 = 0$, $a_1 = 1$.

(b) A *divide and conquer* algorithm works on an integer array of size $n$. For $n \geq 2$, it divides the array into two almost equal halves and *recursively* processes each part. After the *recursive calls* return, it takes constant time 1 (i.e., just one elementary operation) to combine the solutions on the parts. Formulate a recurrence for the time complexity function $t(n)$ and use *induction* on $n$ to show that $t(n) \in O(n)$. Note that $n$ is any positive integer $\geq 2$ and not necessarily a power of 2.

[8 + 6 = 14]

6. (a) Let $T(r, n)$ be the number of *onto functions* from a set with cardinality $r$ to a set with cardinality $n$. Prove the following recurrence using *combinatorial argument*:

$$T(r, n) = nT(r - 1, n - 1) + nT(r - 1, n).$$

*Algebraic derivation using any explicit formula would lead to zero credit.*

(b) Prove that the number of *p-partitions* of a positive integer $n$ is equal to the number of *partitions* of $n + \binom{p}{2}$ into $p$ *distinct* parts.

[4 + 4 = 8]