Discrete Mathematics 13/02/2019

Lecture 9: Integer Partitions

Instructor: Goutam Paul *Scribe: Sayak Chatterjee*

Contents

9.1 Introduction

In number theory and combinatorics, a partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers.

For example, the number 5 can be represented as a sum of positive integers in 7 distinct ways which are given by:

- 5
- 4+1
- 3+2
- 3+1+1
- 2+2+1
- 2+1+1+1
- 1+1+1+1+1

Observe that we are not considering $4 + 1$ and $1 + 4$ as two different partitions, that is, the order of the parts does not matter here. Now let us formally define the concept of partitions.

Definition 9.1 (p-Partition)**.** For $n \in \mathbb{N}$, a p-partition of n is defined by a *multiset* $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ such that each λ_i is a positive integer and

$$\sum_{i=1}^{p} \lambda_i = n.$$

Here *multiset* is a modification of the concept of set, which allow more than one occurrence of elements unlike a set.

Here we are writing a p-partition with set notations, but some of the elements may occur more than once. To avoid this ambiguity, we will give an alternative definition of a p-partition.

For $n \in \mathbb{N}$, a p-partition of n is defined by an ordered *p-tuple* or a vector $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ where each λ_i is a positive integer and satisfy $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ and

$$\sum_{i=1}^{p} \lambda_i = n$$

.

Example: The partitions of 5, given above are classified as follows:

- (5) is a 1-partition of 5.
- (4,1), (3,2) are 2-partitions of 5.
- (3,1,1), (2,2,1) are 3-partitions of 5.
- (2,1,1,1) is a 4-partition of 5.
- (1,1,1,1,1) is a 5-partition of 5.

Definition 9.2 (Partition)**.** For $n \in \mathbb{N}$, a partition of n is an ordered *tuple* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ and

$$\sum_{i=1}^{p} \lambda_i = n.$$

Here p may vary from 1 to n since we must have at least one part in each partition and number of parts in a partition of n cannot exceed n.

Definition 9.3. For $n, p \in \mathbb{N}$ and $1 \leq p \leq n$, define

$$\pi_p^n = \text{Number of } p\text{-partitions of } n.$$

Example:

$$\pi_1^5 = 1,\ \pi_2^5 = 2,\ \pi_3^5 = 2,\ \pi_4^5 = 1,\ \pi_5^5 = 1.$$

Definition 9.4. For $n \in \mathbb{N}$, define

$$P(n) = \text{Number of partitions of } n = \sum_{p=1}^{n} \pi_p^n.$$

Example:

$$P(5) = \sum_{p=1}^{5} \pi_p^5 = \pi_1^5 + \pi_2^5 + \pi_3^5 + \pi_4^5 + \pi_5^5 = 1 + 2 + 2 + 1 + 1 = 7.$$

Definition 9.5. For $n \in \mathbb{N}$, define

$$Q(n) = \text{Number of partitions of } n \text{ into distinct parts.}$$

In other words, if $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a p-partition of n, we count it in $Q(n)$ if all the λ_i's are distinct.

Example: Observe that (5), (4,1), (3,2) are the only partitions of 5 in which all the parts are distinct. Hence $Q(5) = 3$.

Definition 9.6. For $n \in \mathbb{N}$, define

$$O(n) = \text{Number of partitions of } n \text{ into odd parts.}$$

In other words, if $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a p-partition of n, we count it in $O(n)$ if all the λ_i's are odd.

Example: Observe that (5), (3,1,1), (1,1,1,1,1) are the only partitions of 5 in which every part is an odd number. Hence $O(5) = 3$.

Definition 9.7. For $n \in \mathbb{N}$, define

$$E(n) = \text{Number of partitions of } n \text{ into even parts.}$$

In other words, if $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a p-partition of n, we count it in $E(n)$ if all the λ_i's are even.

Remark 9.8. Observe that for an odd natural number n, we cannot divide into positive integer parts so that all the parts are even numbers, because sum of even numbers is always even. Hence for any odd n, we have $E(n) = 0$.

Example: From the remark, it is clear that $E(5) = 0$. For $n = 4$, we observe that (4) and (2,2) are the only partitions in which every part is an even number. Hence $E(4) = 2$.

9.2 Some Properties of Integer Partitions

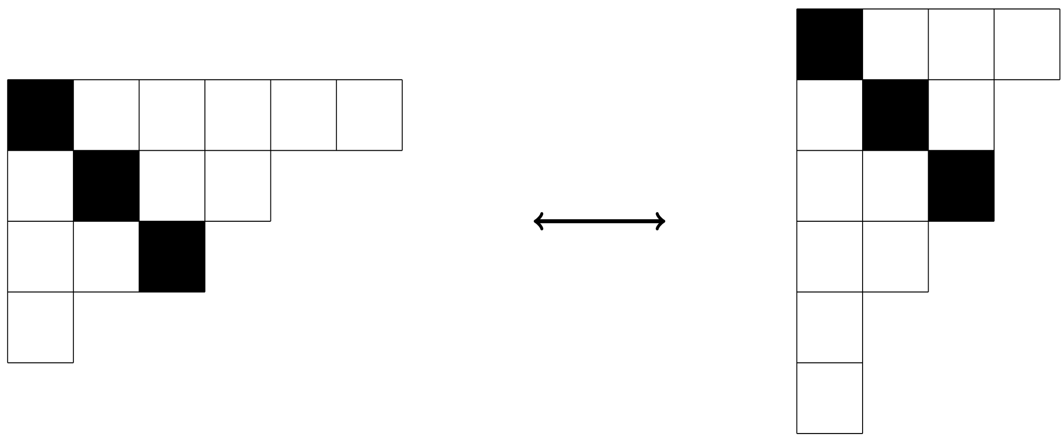
Now let us investigate some properties of integer partitions. Let us take $n = 14$ and consider the partition (6,4,3,1) of 14. First we write each part of the partition horizontally using dots and stack the parts as rows. Then we flip the arrangement along the main diagonal (represented by black dots) to get another partition of 14 namely (4,3,3,2,1,1).

$$\begin{array}{cclllllll} 6 & \rightarrow & \bullet & \circ & \circ & \circ & \circ & \circ \\ 4 & \rightarrow & \circ & \bullet & \circ & \circ & & \\ 3 & \rightarrow & \circ & \circ & \bullet & & & \\ 1 & \rightarrow & \circ & & & & & \end{array} \quad \Longleftrightarrow \quad \begin{array}{lllll} \bullet & \circ & \circ & \circ & \rightarrow 4 \\ \circ & \bullet & \circ & & \rightarrow 3 \\ \circ & \circ & \bullet & & \rightarrow 3 \\ \circ & \circ & & & \rightarrow 2 \\ \circ & & & & \rightarrow 1 \\ \circ & & & & \rightarrow 1 \end{array}$$

Remark 9.9. When we represent a partition of n in the above way, that is, using dots horizontally to represent the parts and stack them vertically to give the whole partition, it is called Ferrers diagram and when the dots are replaced by squares, it is called Young diagram. These are two common diagrammatic representation of integer partitions. The Young diagram of the partitions

$$6 + 4 + 3 + 1 = 4 + 3 + 3 + 2 + 1 + 1$$

of 14 are depicted as follows:



Definition 9.10 (Conjugate Partitions)**.** For $n \in \mathbb{N}$, if the Ferrers (or Young) diagrams of two partitions are just reflections of each other about the main diagonal of the diagram (represented by black dots or black squares), then these two partitions of n are called conjugate (or transpose) partitions of each other.

Example: We can clearly see that (6,4,3,1) and (4,3,3,2,1,1) are two conjugate partitions of 14. We will use this idea of conjugate partitions in the following result.

Theorem 9.11. For $n, p \in \mathbb{N}$ and $1 \leq p \leq n$,

Number of p-partitions of $n = \pi_p^n =$ Number of partitions of n with maximum part p.

Proof. Fix $n \in \mathbb{N}$. Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a p-partition of n. Define the $p \times \lambda_1$ matrix P_Λ as follows. The first row consists of all λ_1 entries as 1. The first λ_2 entries of the second row are 1 and the remaining entries of the second row are zero. In general, in the kth row, the first λ_k entries are 1 and the remaining entries of that row are zero (for $k = 1, 2, \ldots, p$). Observe that the row sums of P_Λ constitute the partition Λ. We can visualize the matrix P_Λ if we replace the dots in Ferrers diagram of Λ by 1 and complete that matrix by putting 0 in the "empty spaces". Now, we use the idea of conjugate partitions. Consider the matrix P_Λ^T. It is of dimension $\lambda_1 \times p$. By construction, the column sums of P_Λ were non-decreasing. Hence the row sums of P_Λ^T will be non-decreasing. Denote the row sums of P_Λ^T by $\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1}$. Then clearly $(\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$ is another partition of n which will be the conjugate partition of Λ. In the first column of P_Λ, all entries were 1 as each of the parts were positive.

So the first column sum of $P_\Lambda = \lambda_1' = p$. Hence we get $(\lambda_1', \lambda_2', \ldots, \lambda_{\lambda_1}')$ as a partition of n whose maximum part λ_1' is p.

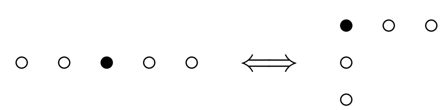
Now if we start with a partition of n whose maximum part is p, the transformation described above again returns a unique p-partition of n which will be the conjugate partition of the one we started with. Thus, there is a one-to-one correspondence between the p-partitions of n and the partitions of n having maximum part p. It completes the proof. □

Definition 9.12 (Self-conjugate Partition)**.** A partition of n for $n \in \mathbb{N}$, is called a self-conjugate partition if it is the conjugate of itself.

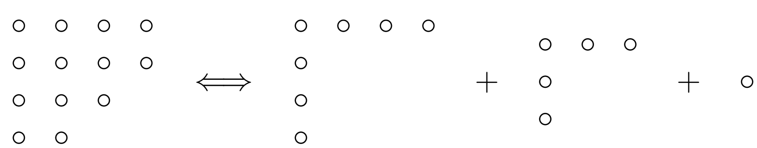
From the definition, it is clear that the diagrams of the self-conjugate partitions should be symmetric about the main diagonal of the diagram. For example, drawing the diagram of the partition (4,2,1,1) of 8, one can easily verify that it is a self-conjugate partition of 8. Now we state another beautiful result in this context.

Theorem 9.13. For $n \in \mathbb{N}$, the number of self-conjugate partitions of n (It is denoted by $S(n)$) is equal to the number of partitions of n into odd parts in which, each part is distinct (It is denoted by $K(n)$).

Proof. Here we will give a sketch of the proof. The main idea behind this proof is: in an odd partition, if we take one of the odd parts, it can be "folded" about its midpoint to form an "L-shape". And these "L-shapes", when taken together, will form a self-conjugate partition.



For $n = 13$ observe that



(4,4,3,2) is a self-conjugate partition of 13 and (7,5,1) is a partition of 13 using distinct odd parts.

In this way, given any self-conjugate partition of n, we can divide it into such "L-shapes". Observe that each "L-shape" has odd number of dots and each of the 'L-shapes' is larger than the next one which is nested inside it. Thus, we get a partition of n with odd distinct parts. On the other hand, given any partition of n with odd distinct parts, we can form such "L-shapes" and nest them together to form a symmetric Ferrers diagram or rather a self-conjugate partition. So we have a bijection between the set of self-conjugate partitions and the set of partitions with odd distinct parts of n. □

9.3 Partitions Using Generating Functions

Suppose we want to calculate π_p^n. This problem also can be viewed as distribution of n many ones into p many parts. We know that in case of p-partitions, order of the parts does not matter. But if we consider ordered partitions, the problem is simplified. It just turns out to be the occupancy problem of placing n balls into p cells where the balls are identical and the cells are distinguishable and each cell should be non-empty.

For this problem, since each cell should be non-empty, firstly we place one ball into each of the p cells. Then the problem reduces to distributing remaining $n-p$ balls into p cells where empty cells are allowed. The number of ways in which this can be done is $C^R(p, n-p) = \binom{n-1}{n-p}$.

We can also solve the same problem using generating functions. Each of the p cells must contain at least one ball and the total number of balls would be n. So number of such distributions equals the coefficient of x^n in the expansion of:

$$\underbrace{(x+x^2+x^3+\ldots)(x+x^2+x^3+\ldots)\ldots(x+x^2+x^3+\ldots)}_{\text{p times}} = (x+x^2+x^3+\ldots)^p$$

$$= \left(\frac{x}{1-x}\right)^p = x^p(1-x)^{-p}$$

which is equivalent to coefficient of x^{n-p} in the expansion of $(1-x)^{-p}$. One can check that this is exactly equal to $\binom{n-1}{n-p}$.

We want to apply this method using generating functions for counting π_p^n, that is when order of the parts does not matter.

From Theorem 9.11, we have $\pi_p^n =$ Number of partitions of n with maximum part p. It would be easier to count the number of partitions of n with maximum part p using generating functions. In the partition, the number of ones used can be any non-negative integer. The power of x in the factor $(1+x+x^2+\ldots)$ denotes how many times 1 is included in the partition. Similarly, the number of twos used can be any non-negative integers and the power of x^2 in the factor $(1+x^2+x^4+\ldots)$ denotes how many times 2 is included in the partition. Up to $(p-1)$, the number of choices can be any non-negative integers. But for p, we have to use it at least once in the whole partition. So for p, the number of choices would be a positive integer. Number of times p is used in the partition is denoted by the power of x^p in the factor $(x^p + x^{2p} + x^{3p} + \ldots)$. Hence π_p^n can be given as the coefficient of x^n in the expansion of:

$$(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots(1+x^{p-1}+x^{2(p-1)}+\ldots)(x^p+x^{2p}+x^{3p}+\ldots)$$

$$= x^p(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots(1+x^p+x^{2p}+x^{3p}+\ldots)$$

$$= \frac{x^p}{(1-x)(1-x^2)\ldots(1-x^p)}.$$

Hence the generating function of $\{\pi_p^n\}_{n\in\mathbb{N}}$ is given by:

$$\Pi_p(x) = \sum_{n=1}^{\infty}(\pi_p^n)x^n = \frac{x^p}{(1-x)(1-x^2)\dots(1-x^p)} = \prod_{k=1}^{p}\left(\frac{x}{1-x^k}\right).$$

Unfortunately, π_p^n does not have a closed form. But still we can have some interesting results.

Notation.

$$\pi_{\le p}^n = \sum_{i=1}^{p}\pi_p^n.$$

Theorem 9.14.

$$\pi_p^n = \pi_{\le p}^{n-p} = \pi_{p-1}^{n-1} + \pi_p^{n-p}.$$

Proof. First we will prove the equality

$$\pi_p^n = \pi_{\le p}^{n-p} \tag{1}$$

using a combinatorial argument. π_p^n can be thought as the number of distributing n identical balls into p identical cells so that no cell is kept empty. So in the distribution procedure, first we put one ball in each of the p cells. This can be done in one way. Then we have $(n-p)$ balls remaining which are to be distributed without any restriction except that the number of parts cannot exceed p. Which is exactly $\pi_{\le p}^{n-p}$. It proves equation 1, which is the first part of the theorem.

The second part can be proved using two ways. Firstly,

$$\begin{aligned}
\pi_p^n = \pi_{\le p}^{n-p} &= \sum_{i=1}^{p}\pi_i^{n-p} \\
&= \pi_1^{n-p} + \pi_2^{n-p} + \dots + \pi_{p-1}^{n-p} + \pi_p^{n-p} \\
&= (\pi_1^{(n-1)-(p-1)} + \pi_2^{(n-1)-(p-1)} + \dots + \pi_{p-1}^{(n-1)-(p-1)}) + \pi_p^{n-p} \\
&= \left(\sum_{i=1}^{p-1}\pi_i^{(n-1-(p-1)}\right) + \pi_p^{n-p} \\
&= \pi_{\le(p-1)}^{(n-1)-(p-1)} + \pi_p^{n-p} = \pi_{p-1}^{n-1} + \pi_p^{n-p} \qquad \text{[By equation 1.]}
\end{aligned}$$

Secondly, using combinatorial argument. A p-partition of n can be of two types: either it may contain at least one partition as 1 or all the parts of the partition is greater than 1. In the former case, 1 is a part of the partition. So we have to divide the rest $(n-1)$ into $(p-1)$ parts. It can be done in π_{p-1}^{n-1} many ways. In the later case, each of the p parts of the partition is greater than 1. So first put all the p parts as 1. After that each of the p parts again will get some positive number of ones (since each of the parts is greater than 1). So we have to divide the remaining $(n-1)$ ones into exactly p parts. It can be done in π_p^{n-p} many ways. Since these two cases are disjoint, the result follows. □

Now let us try to find the generating function of $P(n)$, that is, number of partitions of n. Observe that, the maximum part can be n. Thus, $P(n)$ can be given by the coefficient of x^n in the expansion of

$$(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)\ldots(1 + x^n + x^{2n} + \ldots)$$
$$= \frac{1}{(1-x)(1-x^2)\ldots(1-x^n)}.$$

Hence the generating function of $\{P(n)\}_{n\in\mathbb{N}}$ (Denoted by P_{gf}) can be given as:

$$P_{gf}(x) = \sum_{n=1}^{\infty} P(n)x^n = \frac{1}{(1-x)(1-x^2)\ldots(1-x^n)} = \prod_{k=1}^{n}\left(\frac{1}{1-x^k}\right).$$

We can also find the generating functions of $Q(n)$, $O(n)$, $E(n)$, $K(n) = S(n)$ (By Theorem 9.13) etc. Let us start with $Q(n)$. It denotes number of partitions of n with distinct parts. So each part can either be used only once or never used. Hence $Q(n)$ would be the coefficient of x^n in the expansion of:

$$(1 + x)(1 + x^2)(1 + x^3)\ldots.$$

Thus, generating function of $\{Q(n)\}_{n\in\mathbb{N}}$ is given by:

$$Q_{gf}(x) = \sum_{n=1}^{\infty} Q(n)x^n = (1 + x)(1 + x^2)(1 + x^3)\ldots = \prod_{k=1}^{\infty}(1 + x^k)$$
$$= \prod_{k=1}^{\infty}\left(\frac{1-x^{2k}}{1-x^k}\right) = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}$$

[The terms $(1 - x^k)$ in denominator with even k would get cancelled.]

For $O(n)$ we have to use only odd numbers as the parts of a partition. Hence $O(n)$ is the coefficient of x^n in the expansion of:

$$(1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x^5 + x^{10} + \ldots)\ldots.$$

Hence, generating function for $\{O(n)\}_{n\in\mathbb{N}}$ is given by:

$$O_{gf}(x) = \sum_{n=1}^{\infty} O(n)x^n = (1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)\ldots$$
$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots} = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}$$

which is exactly equal to the expression of $Q_{gf}(x)$. Thus in both $Q_{gf}(x)$ and $O_{gf}(x)$, the coefficient of x^n must match. Hence we get a surprising result:

Theorem 9.15. For $n \in \mathbb{N}$, we have $Q(n) = O(n)$.

We will give a combinatorial proof of this as well.

Proof. Start with an odd partition of n, that is, partition in which each part is odd. Suppose in that partition, 1 occurs t_1 times, 3 occurs t_3 times, 5 occurs t_5 times, and so on. Thus $n = t_1 \times 1 + t_3 \times 3 + t_5 \times 5 + \dots$. Now write each of the t_i's using their binary expansion:

$$n = (2^{k_{11}} + 2^{k_{12}} + \dots)1 + (2^{k_{31}} + 2^{k_{32}} + \dots)3 + (2^{k_{51}} + 2^{k_{52}} + \dots)5 + \dots.$$

Now if we just get rid of the brackets, observe that the terms in the sum would be distinct. Because if $2^a b = 2^c d$, where a, b, c, d are non-negative integers and b, d are odd, then one can easily show that $a = c$ and $b = d$. It shows that starting with an odd partition of n we can reach a unique partition of n with distinct parts.

On the other hand, lets start with a partition of n with distinct parts. We write each part as $\lambda_k = 2^{a_k} \times b_k$ where b_k is odd. That is, we are extracting the powers of 2 from λ_k till an odd integer is left. Now observe that since each b_k is odd, we collect their coefficients and write b_k those many times. Then it will give an odd partition of n. So starting from a partition with distinct parts, we have reached a unique partition of n with odd parts.

So, there is a bijection between the partitions of n with distinct parts and with odd parts. It completes the proof. □

Let us come back to our discussion of generating functions. Generating function of $E(n)$ is derived almost similarly as that of $O(n)$. The only difference is, we have to use even parts here. So the generating function of $\{E(n)\}_{n\in\mathbb{N}}$ is given by:

$$E_{gf}(x) = \sum_{n=1}^{\infty} E(n)x^n = (1 + x^2 + x^4 + \dots)(1 + x^4 + x^8 + \dots)(1 + x^6 + x^{12} + \dots)\dots$$

$$= \frac{1}{(1 - x^2)(1 - x^4)(1 - x^6)\dots} = \prod_{k=1}^{\infty} \frac{1}{1 - x^{2k}}.$$

Similarly, we can get the generating function of $K(n)$ or $S(n)$ (They are same by Theorem 9.13.) which denotes the number of partitions of n with distinct odd parts. The generating function of $\{K(n)\}_{n\in\mathbb{N}}$ is given by:

$$K_{gf}(x) = \sum_{n=1}^{\infty} K(n)x^n = (1 + x)(1 + x^3)(1 + x^5)\dots = \prod_{k=1}^{\infty}(1 + x^{2k-1}).$$

Definition 9.16.

$$Q(n, p) = \text{Number of partitions of } n \text{ with } p \text{ distinct parts.}$$

Now let us find the generating function of $Q(n, p)$.

Proposition 9.17.

$$Q(n, p) = \pi_p^{n-\binom{p}{2}}.$$

Proof. Let us start with a p-partition of n with all distinct parts. Let the partition be $(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Since the parts are distinct, we have $\lambda_1 > \lambda_2 > \ldots > \lambda_p \geq 1$. So $\lambda_p \geq 1$, $\lambda_{p-1} \geq 2$, $\lambda_{p-2} \geq 3$, ..., $\lambda_1 \geq p$. In general, $\lambda_{p-k} \geq k+1$ for $k = 1, 2, \ldots, (p-1)$. Define $\mu_{p-k} = \lambda_{p-k} - k \geq 1$ for $k = 1, 2, \ldots, (p-1)$ and $\mu_p = \lambda_p$. Then we observe that $\mu_1, \mu_2, \ldots, \mu_{p-1}, \mu_p \geq 1$ And since we had $\lambda_1 > \lambda_2 > \ldots > \lambda_{p-1} > \lambda_p \geq 1$, now we have $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{p-1} \geq \mu_p \geq 1$. And

$$\left(\sum_{k=1}^{p} \mu_k\right) = \left(\sum_{k=1}^{p} \lambda_k\right) - (1 + 2 + \ldots + (p-1)) = n - \frac{p(p-1)}{2} = n - \binom{p}{2}.$$

Hence $(\mu_1, \mu_2, \ldots, \mu_{p-1}, \mu_p)$ gives a p-partition of $n - \binom{p}{2}$. The map from a p-partition of n with distinct parts to a p-partition of $n - \binom{p}{2}$ is shown below.

$$(\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \lambda_p) \mapsto (\lambda_1 - (p-1), \lambda_2 - (p-2), \ldots, \lambda_(p-1) - 1, \lambda_p)$$
$$= (\mu_1, \mu_2, \ldots, \mu_{p-1}, \mu_p).$$

Similarly, we can have the inverse of this map: starting from a p-partition of $n - \binom{p}{2}$ to a p-partition of n with distinct parts.

$$(\mu_1, \mu_2, \ldots, \mu_{p-1}, \mu_p) \mapsto (\mu_1 + (p-1), \mu_2 + (p-2), \ldots, \mu_(p-1) + 1, \mu_p)$$
$$= (\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \lambda_p).$$

This bijection completes the proof. □

Now we know that $Q(n,p) = \pi_p^{n+\binom{p}{2}}$ which is given by:

$$\text{Coefficient of } x^{n-\binom{p}{2}} \text{ in } \frac{x^p}{(1-x)(1-x^2)\ldots(1-x^p)} \quad [\text{By generating function of } \pi_p^n]$$

$$= \text{Coefficient of } x^n \text{ in } \frac{x^{p+\binom{p}{2}}}{(1-x)(1-x^2)\ldots(1-x^p)} = \frac{x^{\frac{p(p+1)}{2}}}{(1-x)(1-x^2)\ldots(1-x^p)}.$$

Hence generating function of $\{Q(n,p)\}_{n\in\mathbb{N}}$ is given by:

$$Q_p(x) = \sum_{n=1}^{\infty} Q(n,p)x^n = \frac{x^{\frac{p(p+1)}{2}}}{(1-x)(1-x^2)\ldots(1-x^p)}.$$

9.4 Some Recurrence Relations

Definition 9.18.

$$P(n,p,q) = \text{Number of } p\text{-partitions of } n \text{ with maximum part } q.$$

Definition 9.19.

$$Q(n,p,q) = \text{Number of } p\text{-partitions of } n \text{ with distinct parts and maximum part } q.$$

Analogously define $P(n, p, \leq q)$, $P(n, \leq p, q)$, $P(n, \leq p, \leq q)$ etc.

Now we will state some recurrence relations regarding partitions and try to prove them using combinatorial arguments.

Proposition 9.20.

$$P(n, p, q) = P(n, q, p).$$

Proof. Observe that this is equivalent to Theorem 9.11. □

Proposition 9.21.

$$P(n, p, q) = P(n - q, p - 1, \leq q) = \sum_{k=1}^{q} P(n - q, p - 1, k).$$

Proof. Start with a p-partition of n with maximum part q. So one part of the partition is q and the other $(p - 1)$ parts are less than or equal to q. So let us remove that part q from the partition. Then there will be $(p - 1)$ parts left which will constitute a partition of $(n - q)$ where the maximum part is less than or equal to q.

On the other hand, suppose we start with a $(p - 1)$-partition of $(n - q)$ whose maximum part is less than or equal to q. We add a part q to that partition. Then we will get a p-partition of n with maximum part q.

So there is a one-to-one correspondence between the p-partitions of n with maximum part q and the $(p - 1)$-partitions of $(n - q)$ with maximum part less than or equal to q. It completes the proof. □

Proposition 9.22.

$$Q(n, p, q) = Q(n - q, p - 1, < q) = \sum_{k=1}^{q-1} Q(n - q, p - 1, k).$$

Proof. Start with a p-partition of n with distinct parts and maximum part q. So one part of the partition is q and the other $(p - 1)$ parts are strictly less than q. So let us remove that part q from the partition. Then there will be $(p-1)$ parts left which will constitute a $(p-1)$-partition of $(n-q)$ where the maximum part will be strictly less than q.

On the other hand, suppose we start with a $(p - 1)$-partition of $(n - q)$ with distinct parts whose maximum part is strictly less than q. We add a part q to that partition (It is greater than all the other parts and hence different). Then we will get a p-partition of n with distinct parts and maximum part q.

So there is a one-to-one correspondence between the p-partitions of n with distinct parts and maximum part q and the $(p - 1)$-partitions of $(n - q)$ with distinct parts whose maximum part is strictly less than q. It completes the proof. □

Proposition 9.23.

$$P(n, \leq p, \leq n) = P(n + p, p, \leq n + 1).$$

Proof. Observe that the condition $\leq n$ is redundant in $P(n, \leq p, \leq n)$. So it just denotes the number of partitions of n with p or less number of parts. It is nothing but $\pi^n_{\leq p}$ in our previous notation. By equation 1, we have $\pi^n_{\leq p} = \pi^{n+p}_p$ which is number of p-partitions of $(n+p)$. But observe that since we have to break $(n+p)$ into p parts, the parts cannot be greater than $(n+1)$. If the greatest part is at least $(n+2)$, there are remaining $(p-1)$ parts which are at least 1. So their sum becomes at least $(n+p+1)$ which is a contradiction. Hence each part should be less than or equal to $(n+1)$. Thus we get $\pi^{n+p}_p = P(n+p, p, \leq n+1)$ which completes the proof. □

Proposition 9.24.

$$P(n, \leq n, \leq q) = \begin{cases} 1, & \text{if} \quad q = 1 \\ P(n, \leq n, \leq n-1) + 1, & \text{if} \quad n = q > 1 \\ P(n, \leq n, \leq q-1) + P(n-q, \leq n, \leq q), & \text{if} \quad n > q > 1. \end{cases}$$

Proof. Observe that $P(n, \leq n, \leq q)$ denotes the number of partitions of n with maximum part less than or equal to q. Since the number of parts in each partition is anyway $\leq n$, we can drop that condition. Now, number of partitions of n with maximum part less than or equal to q equals number of partitions of n with q or less parts (By Theorem 9.11). So, $P(n, \leq n, \leq q) = \pi^n_{\leq q} = \pi^{n+q}_q$ by equation 1.

$$P(n, \leq n, \leq q) = \pi^{n+q}_q. \tag{2}$$

Now let us consider different cases.

Case 1: $q = 1$. We have to calculate π^{n+1}_1 that is, dividing $(n+1)$ in 1 part which can be done in only one way. So $\pi^{n+1}_1 = 1$.

Case 2: $n = q > 1$. As $n = q$, $\pi^{n+q}_q = \pi^{2n}_n$. By Theorem 9.14, we have:

$$\pi^{2n}_n = \pi^{2n-1}_{n-1} + \pi^n_n = \pi^{n+(n-1)}_{n-1} + 1 \quad [\text{As } \pi^n_n = 1]$$

$$= P(n, \leq n, \leq n-1) + 1 \quad [\text{By equation 2}].$$

Case 3: $n > q > 1$. Again we use Theorem 9.14,

$$\pi^{n+q}_q = \pi^{n+q-1}_{q-1} + \pi^{n+q-q}_q = \pi^{n+(q-1)}_{q-1} + \pi^{(n-q)+q}_q$$

$$= P(n, \leq n, \leq q-1) + P(n-q, \leq n-q, \leq q) \quad [\text{By equation 2}]$$

$$= P(n, \leq n, \leq q-1) + P(n-q, \leq n, \leq q).$$

□