Discrete Mathematics 11/02/2019

Lecture 8: Generating Functions

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Contents

1 Ordinary Generating Functions

1.1 Introduction

In mathematics, a generating function is a way of encoding an infinite sequence of numbers (a_n) by treating them as the coefficients of a power series. This formal power series is the generating function. Unlike an ordinary series, this formal series is allowed to diverge, meaning that the generating function is not always a true function and the "variable" is actually an indeterminate. Generating functions were first introduced by Abraham de Moivre in 1730, in order to solve the general linear recurrence problem. One can generalize to formal series in more than one indeterminate, to encode information about arrays of numbers indexed by several natural numbers.

1.2 Definition

Given any sequence $(a_0, a_1, \ldots, a_n, \ldots)$ the ordinary generating function of that sequence is defined as the "formal" sum

$$G(x) = a_0 + a_1x + a_2x^2 \ldots \tag{1}$$

.
For example if $a_n = 1$
$\forall n \in \mathbb{N}$, then

$$G(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \quad |x| < 1 \tag{2}$$

The ordinary generating function for the infinite sequence $a_0, a_1, a_2, a_3, \ldots$ is the power series:

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \tag{3}$$

Not all generating functions are ordinary, but those are the only kind we'll consider here. A generating function is a "formal" power series in the sense that we usually regard x as a blank rather than assuming some value. Generally we ignore the issue of convergence. The correspondence between a sequence and its generating function will be denoted like this from here on by $\iff$

$$a_0, a_1, a_2, a_3, \ldots \iff a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \tag{4}$$

For example, here are some sequences and their generating functions:

$$(0, 0, 0, 0, ...) \iff 0 + 0x + 0x^2 + 0x^3 + \cdots = 0 \tag{5}$$

$$(1, 0, 0, 0, ...) \iff 1 + 0x + 0x^2 + 0x^3 + \cdots = 1 \tag{6}$$

The pattern here is simple: the i-th term in the sequence (indexing from 0) is the coefficient of x^i in the generating function.
Recall that the sum of an infinite geometric series is:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{7}$$

This equation does not hold when $|x| \geq 1$, but as remarked, we don't worry about convergence issues. This formula gives closed form generating functions for a whole range of sequences. For example:

$$(1, 0, 1, 0, 1, 0, \ldots) \iff 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2} \tag{8}$$

1.3 Facts

If $G(x)$ is the generating function of $a_0, a_1, a_2, ..,$ then;

- $G(x)/(1-x)$ is the generating function of $a_0, a_0+a_1, a_0+a_1+a_2 \ldots.$
- $G'(x)$ is the generating function of $a_1, 2a_2, 3a_3 \ldots.$
- $G(x^k)$ is the generating function of $a_0, \underbrace{0, \ldots, 0}_{\text{(k-1)times}}, a_1, \underbrace{0, \ldots, 0}_{\text{(k-1)times}}, a_2, \ldots.$
- $x^k G(x^k)$ is the generating function of $\underbrace{0, 0, \ldots, 0}_{\text{k-times}}, a_0, a_1, a_2, \ldots.$

From here we realise that there can be many operations on generating functions that can make our lives easier while solving problems using generating functions. For that we have our next section on operations of generating functions.

2 Operations on Generating Functions

We can carry out all sorts of manipulation with the polynomial to give us different interesting results.

2.1 Scalar Multiplication

Multiplying a generating function by a constant scales every term in the associated sequence by the same constant. For example, we noted in (8) . Multiplying the generating function by 2 gives

$$(2, 0, 2, 0, 2, 0, \ldots) \iff 2 + 2x^2 + 2x^4 + 2x^6 + \cdots = \frac{2}{1-x^2} \tag{9}$$

Theorem 2.1 (Scaling Rule)**.** If $(a_0, a_1, a_2, \ldots) \iff F(x)$, then $(ca_0, ca_1, ca_2, \ldots) \iff cF(x)$.

Proof. The idea behind this rule is that: $(ca_0, ca_1, ca_2, \ldots) \iff ca_0 + ca_1x + ca_2x^2 + \ldots$
$= c \cdot (a_0 + a_1x + a_2x^2 + \ldots)$
$= cF(x)$ □

2.2 Adding

Adding generating functions corresponds to adding the two sequences term by term. For example, adding two of our earlier examples gives:

$$(1,1,1,1,1,1,\dots) \iff \frac{1}{1-x} \tag{10}$$

$$(1,-1,1,-1,1,-1,\dots) \iff \frac{1}{1+x} \tag{11}$$

Adding (11) with (10) yields

$$(2,0,2,0,2,0,\dots) \iff \frac{1}{1-x} + \frac{1}{1+x}$$

Over this period of time we got two different expressions that both generate the sequence $(2,0,2,0,\dots)$. They are, of course, equal: $\frac{1}{1+x} + \frac{1}{1-x} = \frac{(1-x)+(1+x)}{(1-x)(1+x)} = \frac{2}{1-x^2}$

Theorem 2.2. (Addition Rule) If $(a_0, a_1, a_2, \dots) \iff F(x)$, and $(b_0, b_1, b_2, \dots) \iff G(x)$, then $(a_0+b_0, a_1+b_1, a_2+b_2, \dots) \iff F(x)+G(x)$.

Proof. The idea behind this rule is that:

$$(a_0+b_0, a_1+b_1, a_2+b_2, \dots) \iff \sum_{n=0}^{\infty}(a_n+b_n)x^n \tag{12}$$

$$= \sum_{n=0}^{\infty}(a_n)\cdot x^n + \sum_{n=0}^{\infty}(b_n)\cdot x^n \tag{13}$$

$$= F(x) + G(x) \tag{14}$$

□

2.3 Right Shifting

We start with a simple sequence and its generating function: $(1,1,1,1,\dots) \iff \frac{1}{1-x}$
Now let's right shift the sequence by adding k leading zeros:

$$(\underbrace{0,\dots,0}_{\text{k-zeroes}},1,1,1,) \iff x^k + x^{k+1} + \cdots = x^k \cdot (1+x+x^2+x^3+\dots) = \frac{x^k}{1-x}$$

Evidently, adding k leading zeros to the sequence corresponds to multiplying the generating function by x^k. This holds true in general.

Theorem 2.3. (Right-Shift Rule) If

$$a_0, a_1, a_2, \dots \iff F(x), \text{then}: (\underbrace{0,0,\dots,0}_{\text{k-zeroes}}, a_0, a_1, a_2, \dots) \iff x^k \cdot F(x)$$

Proof. The idea behind this rule is that:

$$(\underbrace{0, 0, \dots, 0}_{\text{k-zeroes}}, a_0, a_1, a_2, \dots) \iff a_0x^k + a_1x^{k+1} + \dots \tag{15}$$

$$= x^k \cdot (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) \tag{16}$$

$$= x^k \cdot F(x) \tag{17}$$

□

2.4 Left Shifting

To start, let's consider an easy example. Suppose we have that

$F(x)$ is the generating function of $(a_0, a_1, a_2, \dots)$ and we want to find the generating function for the sequence $(a_1, a_2, a_3, \dots)$. Formally, we want the function $G(x)$ that has power series representation

$$G(x) = \sum_{n=0}^{\infty} a_{n+1}x^n \tag{18}$$

Since we know that

$$F(x) = \sum_{n=0}^{\infty} a_nx^n \tag{19}$$

We have that

$$F(x) - a_0 = \sum_{n=1}^{n=\infty} a_nx^n \tag{20}$$

and re indexing gives

$$F(x) - a_0 = \sum_{n=0}^{\infty} a_{n+1}x^{n+1} \tag{21}$$

. Finally, dividing both sides by x, we achieve

$$\frac{F(x) - a_0}{x} = \sum_{n=0}^{\infty} a_{n+1}x^n \tag{22}$$

which was precisely the function we wanted. Setting

$$G(x) = \frac{F(x) - a_0}{x} \tag{23}$$

we now have $G(x)$ is the generating function of $(a_1, a_2, a_3, \dots)$, as desired. From this example, we can derive a rule.

Theorem 2.4. (Left Shift Rule) If $F(x)$ is the generating function of $(a_0, a_1, a_2, \dots)$, then the ordinary generating function of the sequence $(a_k, a_{k+1}, a_{k+3}, \dots)$, is given by

$$G(x) = \frac{F(x) - a_0 - a_1x - \dots - a_{k-1}x^{k-1}}{x^k} \tag{24}$$

2.5 Substitution

Let $F(x)$ be the generating function of $(a_0, a_1, a_2, \ldots)$. Then we have

$$F(x) = \sum_{n=0}^{\infty} a_n x^n \tag{25}$$

Now by substituting αx in place of x , we get

$$F(\alpha x) = \sum_{n=0}^{\infty} a_n (\alpha x)^n \tag{26}$$

$$F(\alpha x) = \sum_{n=0}^{\infty} a_n \alpha^n x^n \tag{27}$$

which is the generating equation of $(a_0, a_1\alpha, a_2\alpha^2, \ldots)$. From this we can conclude

Theorem 2.5. (Substitution Rule If $F(x)$ is the generating function of $(a_0, a_1, a_2, \ldots)$, then $F(\alpha x)$ is the generating function of $(a_0, \alpha a_1, \alpha^2 a_2, \ldots)$.

2.6 Differentiation

In general, differentiating a generating function has two effects on the corresponding sequence: each term is multiplied by its index and the entire sequence is shifted left one place.

Theorem 2.6. (Derivative Rule)

$$(a_0, a_1, a_2, a_3, \ldots) \iff F(x), \text{then } (a_1, 2a_2, 3a_3, \ldots) \iff F'(x).$$

Proof. The idea behind this rule is that:

$$(a_1, 2a_2, 3a_3, \ldots) \iff a_1 + 2a_2x + 3a_3x^2 + \ldots = \frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots) \tag{28}$$

$$= \frac{d}{dx}F(x) \tag{29}$$

□

The Derivative Rule is very useful. It is used along with the other tools.

2.7 Integration

Let $f(x)$ be the generating function of $a_0, a_1, a_2, \ldots$, then we have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{30}$$

If this formal sum is uniformly convergent, we can integrate term by term to get

$$F(x) = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \tag{31}$$

where $F(x)$ is the function obtained by integrating $f(x)$. Note that the sum in RHS is the generating function of $(0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots)$. This is an useful method for finding the generating function of a sequence.

2.8 Products

One can also try to take products of two generating functions. That proves to be helpful in quite a lot of calculations.

Theorem 2.7. (Product Rule). If

$$(a_0, a_1, a_2, \ldots) \iff A(x),$$

and

$$(b_0, b_1, b_2, \ldots) \iff B(x),$$

then

$$(c_0, c_1, c_2, \ldots) \iff A(x) \times B(x),$$

where

$$c_n := a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.$$

Proof. To understand this rule, let

$$C(x) := A(x) \times B(x) = \sum_{(n=0)}^{\infty} c_n x^n.$$

We can evaluate the product $A(x) \times B(x)$. Notice all terms involving the same power of x . Collecting these terms together, we find that the coefficient of x^n in the product is

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0. \tag{32}$$

□

This expression (32) may be familiar from *Analysis I* course, the sequence $(c_0, c_1, c_2, \ldots)$ is the convolution of sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$.

2.9 Convolution Rule

Let $A(x)$ be the generating function for selecting items from set A, and let $B(x)$ be the generating function for selecting items from set B. If A and B are disjoint, then the generating function for selecting items from the union $A \cup B$ is the product $A(x) \times B(x)$.

This rule is rather ambiguous: what exactly are the rules governing the selection of items from a set? Convolution Rule remains valid under many interpretations of selection. We will see that in the examples.

Example 2.8. Find generating function of the sum: $\sum_{k=0}^{n}(n-k)a_k$,

Proof.

$$\left(\sum_{n\geq 0} nx^n\right)\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}(n-k)a_k\right)x^n .$$

We know that

$$\frac{x}{(1-x)^2} = \sum_{n\geq 0} nx^n ,$$

so if g is the generating function for $\langle a_n : n \in \mathbb{N}\rangle$, then $\sum_{k=0}^{n}(n-k)a_k$ is the coefficient of x^n in the power series expansion of

$$\frac{xg(x)}{(1-x)^2} \tag{33}$$

If $g(x)$ is a rational function, expand Equation (33) into partial fractions, expand in power series, and combine to get the desired coefficient. □

Example 2.9. (Catalan Numbers)
Let us consider a counting problem. Suppose we want to find out the number of monotonic lattice paths of length $2n$ in an $n \times n$ grid that do not rise above the main diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards.Let this number be C_n. Let X stand for "move right" and Y stands for "move up". We want to find number of strings consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.
In the above problem think of the X's as open parenthesis and Y's as closed parenthesis. Then the required number is same as the number of valid parenthesis for $n+1$ terms.
Observe that C_n described above, follows the recurrence

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

with initial value $C_0 = 1$. This is true indeed as we can find one valid parenthesizing by parenthesizing the first k terms and then the next $n-k$ terms for any k between 0 and $n-1$.
Now consider a convex polygon with $n+2$ sides. This polygon can be cut into triangle by connecting vertices with non crossing line segments.It is easy to see that the number of triangles formed will be n. Now observe that we may first connect two vertices to form two convex polygons with respectively $k+2$ and $n-k+2$ vertices for any k between 0 and $n-1$. (Both polygons share two vertices). Now we have reduced the problem to the cases of forming k and $n-k$ triangles in the first and second polygon respectively. Hence we see this number too follows the above recurrence relation with the same initial value.
Hence, we get that

- For the problem of monotonic lattice paths below the diagonal of length $2n$, the required number is C_n. It is easy to see that for $n = 1$, the number of paths is $1 = C_1$ and for $n = 2$, the number is $2 = C_2$.(The paths are XXYY and XYXY). We take the number to be 1 for $n = 0$. Hence, by recurrence our claim follows.

- For the problem of parenthesizing, the required number for parenthesizing n items is C_{n-1}, for $n > 1$. We can check that for $n = 2$, the number is $1 = C_1$ and for $n = 3$ the number is $2 = C_2$. For the case of 1 item, we take the number to be 1.

- For the problem of a convex polygon with n vertices, The number of ways to divide it into $n-2$ triangles is C_{n-2}, for $n > 2$. Here also, we see that for $n = 3$, the polygon is a triangle itself, so number of ways is $1 = C_1$ and for $n = 4$, the polygon is a convex quadrilateral, in which we can join any of the 2 diagonals. So required number is $2 = C_2$.

These numbers C_n are known as the Catalan numbers and we see them in many counting problems as in a few problems described above. Now we will try and find the value of C_n using generating functions.

Proof. Let the ordinary generating function be

$$c(x) = \sum_{n\geq 0} C_n x^n = \sum_{n\geq 0} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n \ .$$

Then since $C_0 = 1$, we have

$$\begin{aligned}
c(x) &= \sum_{n\geq 0}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n \\
&= 1 + \sum_{n\geq 1}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n \\
&= 1 + x\sum_{n\geq 0}\sum_{k=0}^{n} C_k C_{n-k} x^n \\
&= 1 + x\left(\sum_{n\geq 0} C_n x^n\right)^2 \\
&= 1 + xc(x)^2 \ ,
\end{aligned}$$

or $xc(x)^2 - c(x) + 1 = 0$. The quadratic formula then yields

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x} \tag{34}$$

and since

$$\lim_{x\to 0} c(x) = \lim_{x\to 0}\sum_{n\geq 0} C_n x^n = C_0 = 1 \ ,$$

it's clear that we must choose the negative square root in Equation (34), so that

$$c(x) = \frac{1 - \sqrt{1-4x}}{2x} \ .$$

Now apply the binomial theorem to $\sqrt{1-4x}$

$$
\begin{aligned}
(1-4x)^{1/2} &= \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n \\
&= \sum_{n\geq 0} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!} (-4x)^n \\
&= \sum_{n\geq 0} (-1)^{n-1} \frac{(2n-3)!!}{2^n n!} (-4x)^n \\
&= -\sum_{n\geq 0} \frac{2^n (2n-3)!!}{n!} x^n \\
&= -2\sum_{n\geq 0} \frac{2^{n-1} \prod_{k=1}^{n-1}(2k-1)}{n(n-1)!} x^n \\
&= -2\sum_{n\geq 0} \frac{2^{n-1}(n-1)! \prod_{k=1}^{n-1}(2k-1)}{n(n-1)!^2} x^n \\
&= -2\sum_{n\geq 0} \frac{\left(\prod_{k=1}^{n-1}(2k)\right)\left(\prod_{k=1}^{n-1}(2k-1)\right)}{n(n-1)!^2} x^n \\
&= -2\sum_{n\geq 0} \frac{(2n-2)!}{n(n-1)!^2} x^n \\
&= -2\sum_{n\geq 0} \frac{1}{n} \binom{2(n-1)}{n-1} x^n \ ,
\end{aligned}
$$

where the constant term is 1 and therefore the constant term in the summation is actually $-\frac{1}{2}$. Thus,

$$
\begin{aligned}
c(x) &= \frac{1}{2x}\left(1 + 2\left(-\frac{1}{2} + \sum_{n\geq 1} \frac{1}{n}\binom{2(n-1)}{n-1} x^n\right)\right) \\
&= \sum_{n\geq 1} \frac{1}{n}\binom{2(n-1)}{n-1} x^{n-1} \\
&= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n \ ,
\end{aligned}
$$

and we have the familiar closed form $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$. □

3 Applications of Generating Function

Generating functions are particularly useful for solving counting problems. In particular, problems involving choosing items from a set often lead to nice generating functions by letting the coefficient of x^n be the number of ways to choose n item.

3.1 Finding Sum of Series

Example 3.1.

$$1^3 + 2^3 + \cdots = \sum_{k=1}^{n} k^3$$

Try to find the sum of the above series using this-

$$s_n = \sum_{k=0}^{n} k^3$$

Proof. Let $s_n = \sum_{k=0}^{n} k^3$; your generating function for these numbers will be

$$f(x) = \sum_{n\geq 0} s_n x^n \ .$$

You know that the sequence satisfies the recurrence $s_n = s_{n-1} + n^3$. Multiply this recurrence by x^n and sum over $n \geq 0$:

$$\sum_{n\geq 0} s_n x^n = \sum_{n\geq 0} s_{n-1} x^n + \sum_{n\geq 0} n^3 x^n.$$

The lefthand side of above is $f(x)$. We assume that $s_n = 0$ for all $n < 0$, so we can rewrite above as

$$f(x) = x \sum_{n\geq 0} s_{n-1} x^{n-1} + \sum_{n\geq 0} n^3 x^n = x \sum_{n\geq 0} s_n x^n + \sum_{n\geq 0} n^3 x^n = x f(x) + \sum_{n\geq 0} n^3 x^n$$

and see that

$$f(x) = \frac{1}{1-x} \sum_{n\geq 0} n^3 x^n \ .$$

To deal with the summation, start with

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n \ .$$

Differentiate and multiply by x to get

$$\frac{x}{(1-x)^2} = \sum_{n\geq 0} n x^n \ .$$

Repeat:

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n\geq 0} n^2 x^n .$$

And one more time:

$$\frac{x(1+4x+x^2)}{(1-x)^4} = \sum_{n\geq 0} n^3 x^n .$$

Thus,

$$f(x) = \frac{x+4x^2+x^3}{(1-x)^5} .$$

Now decompose f into partial fractions:

$$f(x) = -\frac{1}{(1-x)^2} + \frac{7}{(1-x)^3} - \frac{12}{(1-x)^4} + \frac{6}{(1-x)^5} .$$

Finally, you need to know some standard generating functions. In particular, you need to know that

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n .$$

With that you get finally that

$$\begin{aligned} f(x) &= \sum_{n\geq 0} \left(-\binom{n+1}{1} + 7\binom{n+2}{2} - 12\binom{n+3}{3} + 6\binom{n+4}{4} \right) x^n \\ &= \sum_{n\geq 0} \frac{1}{4}\left(n^4+2n^3+n^2\right) x^n \end{aligned}$$

and therefore that

$$s_n = \frac{1}{4}\left(n^4+2n^3+n^2\right) = \left(\frac{n(n+1)}{2}\right)^2$$

□

3.2 Solving Recurrence Relations

Example 3.2. $a_n = 2a_{n-1}, a_0 = 1$. Find the general term.

Proof.

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

or,

$$G(x) - a_0 = 2a_0 x + 2a_1 x^2 + \cdots = 2xG(x)$$

or,

$$G(x) = \frac{1}{1-2x}$$

or,

$$G(x) = 1 + 2x + 4x^2 + \cdots = \sum_{n=0}^{\infty} 2^n x^n$$

$$\implies a_n = 2^n$$

□

Example 3.3. $a_n = 2a_{n-1} + 4^{n-1}, a_0 = 1$. Find the general term.

Proof. Multiply both sides of the given relation by x^{n-1} and sum over from $n = 1$ to get,

$$\sum_{n=1}^{\infty} a_n x^{n-1} = 2\sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} (4x)^{n-1}$$

or,

$$\frac{G(x) - a_0}{x} = 2G(x) + \frac{1}{1-4x}$$

or,

$$G(x) - 2xG(x) = 1 + \frac{x}{1-4x}$$

or,

$$G(x) = \frac{(1-3x)}{(1-2x)(1-4x)} = \frac{1}{2}\left[\frac{1}{(1-2x)} + \frac{1}{(1-4x)}\right]$$

$$\implies a_n = \frac{2^n + 4^n}{2}$$

□

Example 3.4. Solve the equation using generating functions.

$$P_n = 2nP_{n-1} - 10n + 5$$

where, $P_0 = 5$.

Proof. Consider the exponential generating function method.

$$P(t) = \sum_{n=0}^{\infty} P_n \frac{t^n}{n!} = 2\sum_{n=0}^{\infty} P_{n-1} \frac{t^n}{(n-1)!} - 10\sum_{n=0}^{\infty} \frac{t^n}{(n-1)!} + 5\sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{35}$$

$$= 2\sum_{n=1}^{\infty} P_{n-1} \frac{t^n}{(n-1)!} - 10\sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} + 5\,e^t \tag{36}$$

$$= 2tP(t) - 10t\,e^t + 5\,e^t = 2tP(t) + 5(1-2t)\,e^t \tag{37}$$

This leads to $P(t) = 5e^t$ or $P_n = 5$. A verification can be made to check that it is indeed a solution. □

Example 3.5. How many ways are there to hand out 24 cookies to 3 children so that they each get an even number, and they each get at least 2 and no more than 10? Use generating functions.

Proof. It's convenient to use the *coefficient of* operator $[x^n]$ to denote the coefficient of x^n of a series.

We obtain

$$[x^{24}](x^2 + x^4 + x^6 + x^8 + x^{10})^3$$

- In here an even number of at least 2 and at most 10 cookies is to distribute to three children. The *coefficient of* operator selects the coefficient of x^{24} which gives the required number.

$$[x^{24}]x^6(1 + x^2 + x^4 + x^6 + x^8)^3 = [x^{18}] \left(\frac{1 - x^{10}}{1 - x^2} \right)^3 .$$

- In above we use the rule

$$[x^{p+q}]A(x) = [x^p]x^{-q}A(x)$$

$$[x^{18}](1 - x^{10})^3 \sum_{n=0}^{\infty} \binom{-3}{n} (-x^2)^n = [x^{18}](1 - 3x^{10}) \sum_{n=0}^{\infty} \binom{n+2}{2} x^{2n}.$$

We expand $(1 - x^{10})^3$ and take the first two terms only since all other terms have powers greater than 18. We also use the binomial identity

$$\binom{-p}{q} = \binom{p+q-1}{p-1} (-1)^q$$

$$\left([x^{18}] - 3[x^8]\right) \sum_{n=0}^{\infty} \binom{n+2}{2} x^{2n} = \binom{11}{2} - 3\binom{6}{2} = 10.$$

□

Example 3.6. Fibonacci Numbers
Consider the sequence of numbers given by

$$F_{n+1} = F_n + F_{n-1}, n > 0 \tag{38}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. We will prove that the general term of this sequence is given by

$$F_n = \frac{1}{\sqrt{5}}\frac{(1+\sqrt{5})^n}{2^n}) - \frac{1}{\sqrt{5}}\frac{(1-\sqrt{5})^n}{2^n} \tag{39}$$

Proof. We have

$$F_{n+1} = F_n + F_{n-1}, n > 0 \tag{40}$$

Let $G(x)$ be the generating function of $(F_0, F_1, \dots)$. Then we have

$$G(x) = \sum_{n=0}^{n=\infty} F_n x^n \tag{41}$$

or,

$$G(x) = 0 + x + \sum_{n=1}^{n=\infty} F_{n+1} x^{n+1} \tag{42}$$

$$G(x) = x + \sum_{n=1}^{n=\infty} F_n x^{n+1} + \sum_{n=1}^{n=\infty} F_{n-1} x^{n+1} \tag{43}$$

$$G(x) = x + x \cdot \sum_{n=1}^{n=\infty} F_n x^n + x^2 \cdot \sum_{n=1}^{n=\infty} F_{n-1} x^{n-1} \tag{44}$$

or

$$G(x) = x + xG(x) + x^2 G(x) \tag{45}$$

or

$$G(x) = \frac{x}{1 - x - x^2} \tag{46}$$

Let

$$\frac{x}{1-x-x^2} = \frac{a}{1-Ax} + \frac{b}{1-Bx} \tag{47}$$

Solving, we get $a = \frac{1}{\sqrt{5}}, b = -\frac{1}{\sqrt{5}}, A = \frac{1+\sqrt{5}}{2}, B = \frac{1-\sqrt{5}}{2}$. Hence, the coefficient of x^n becomes

$$F_n = \frac{1}{\sqrt{5}}\frac{(1+\sqrt{5})^n}{2^n}) - \frac{1}{\sqrt{5}}\frac{(1-\sqrt{5})^n}{2^n} \tag{48}$$

□

3.3 Counting

3.3.1 Combinations (Unordered Selections)

Consider,

$$(1+ax)(1+bx)(1+cx) = 1 + (a+b+c)x + (ab+bc+ca)x^2 + abcx^3$$

- **Theorem 3.7.** If we want to choose r objects out of n objects without repetition, then we need to check the coefficient of x^r in $(1+x)^n = \binom{n}{r}$

  Proof: It is same as selecting r many factors from

$$\underbrace{(1+x)\times(1+x)\cdots\times(1+x)}_{\text{n-factors}}$$

- **Theorem 3.8.** To distribute n identical objects to r distinct people we look at the coefficient of x^n in $(1+x+x^2+\dots)^r = \binom{n+r-1}{r}$

  Proof: Now we have different powers of x here.So we can chose as many powers as we want,and they will sum up to n.

- To distribute n identical objects to r distinct people with a maximum cap of say k many objects per person,we look at $(1+x+x^2+\cdots+x^k)^r$.We can generalize it for different upper caps also.

- We can also introduce a lower cap of l say and now we check the coefficient here.

$$(x^l + x^{l+1} + \dots)^r$$

Example 3.9. Find the generating function where the coefficient a_k represent the how many ways to distribute k pieces of candy to n children where no children get more than m pieces

Proof. This is the number of tuples $(c_1, \dots c_n)$ of non-negative integers such that $\sum c_i = k$ and $c_i \le m$. Each c_i contributes a factor $(1+x+\cdots+x^m)$ to the generating function and the final generating function is

$$\sum a_k x^k = (1+x+\dots x^m)^n.$$

□

Example 3.10. There are three baskets on the ground: one has 2 purple eggs, one has 2 green eggs, and one has 3 white eggs. Eggs of the same color are indistinguishable. In how many ways can I choose 4 eggs from the baskets?

Proof. The generating function for the first basket is $1 + x + x^2$, since there is one way to choose 0 purple eggs (do nothing), 1 way to choose 1 purple egg (since eggs are indistinguishable), and 1 way to choose 2 purple eggs. In a similar fashion the generating function for the green egg basket is $1+x+x^2$ and the generating function for the white egg basket is $1 + x + x^2 + x^3$. Now to find the number of ways to pick eggs from multiple baskets, just simply multiply the functions together, getting $1 + 3x + 6x^2 + 8x^3 + 8x^4 + 6x^5 + 3x^6 + x^7$. We want the number of ways to choose 4 eggs, so we just need to look at the coefficient of x^4 and see that there are 8 ways to choose 4 eggs. □

Example 3.11. The expression

$$(x + y + z)^{2006} + (x - y - z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

Proof. The goal is to find the generating function for the number of unique terms in the simplified expression (in terms of k). In other words, we want to find $f(x)$ where the coefficient of x^k equals the number of unique terms in $(x+y+z)^k + (x-y-z)^k$.

First, we note that all unique terms in the expression have the form, $Cx^ay^bz^c$, where $a + b + c = k$ and C is some constant. Therefore, the generating function for the MAXIMUM number of unique terms possible in the simplified expression of $(x + y + z)^k + (x - y - z)^k$ is

$$(1 + x + x^2 + x^3 \cdots)^3 = \frac{1}{(1 - x)^3}$$

Secondly, we note that a certain number of terms of the form, $Cx^ay^bz^c$, do not appear in the simplified version of our expression because those terms cancel. Specifically, we observe that terms cancel when $1 \equiv b + c \pmod 2$ because every unique term is of the form:

$$\binom{k}{a, b, c} x^ay^bz^c + (-1)^{b+c} \binom{k}{a, b, c} x^ay^bz^c$$

for all possible a, b, c.

Since the generating function for the maximum number of unique terms is already known, it is logical that we want to find the generating function for the number of terms that cancel, also in terms of k. With some thought, we see that this desired generating function is the following:

$$2(x + x^3 + x^5 \cdots)(1 + x^2 + x^4 \cdots)(1 + x + x^2 + x^3 \cdots) = \frac{2x}{(1 - x)^3(1 + x)^2}$$

Now, we want to subtract the latter from the former in order to get the generating function for the number of unique terms in $(x+y+z)^k+(x-y-z)^k$, our initial goal:

$$\frac{1}{(1-x)^3}-\frac{2x}{(1-x)^3(1+x)^2}=\frac{x^2+1}{(1-x)^3(1+x)^2}$$

which equals

$$(x^2+1)(1+x+x^2\cdots)^3(1-x+x^2-x^3\cdots)^2$$

The coefficient of x^{2006} of the above expression equals

$$\sum_{a=0}^{2006}\binom{2+a}{2}\binom{1+2006-a}{1}(-1)^a+\sum_{a=0}^{2004}\binom{2+a}{2}\binom{1+2004-a}{1}(-1)^a$$

Evaluating the expression, we get 1008016. □

3.3.2 Permutations (Ordered Arrangements)

Consider the previous function

$$(1+ax)(1+bx)(1+cx)=1+1!\cdot\frac{(a+b+c)x}{1!}+2!\cdot\frac{(ab+bc+ca)x^2}{2!}+3!\cdot\frac{abcx^3}{3!}$$

.Plug in 1 in place of a,b,c and play around with it to get-

- Permuting r out of n objects without repetition we check the coefficient of $\frac{x^r}{r!}$ in $\left(1+\frac{x}{1!}\right)^n$.
- Permuting r out of n objects with repetition we check the coefficient of $\frac{x^r}{r!}$ in

  $$\left(1+\frac{x^1}{1!}+\frac{x^2}{2!}+\dots\right)^n=e^{nx},$$

  which is the same as n^r.
- Permuting r out of n objects with repetition and with lower(l_i) and upper bound(d_i).We check the coefficient of $\frac{x^r}{r!}$ in

  $$\prod_{i=1}^{n}\left(\frac{x^{l_i}}{l_i!}+\dots+\frac{x^{d_i}}{d_i!}\right)$$

- To permute r out of n objects with repetition and taking at least 1 of each type, we look at coefficient of $\frac{x^r}{r!}$ in

  $$\left(\frac{x^1}{1!}+\frac{x^2}{2!}+\dots\right)^n=(e^x-1)^n=\sum_{i=0}^{n}\binom{n}{i}(-1)^i e^{x(n-i)}=\sum_{i=0}^{n}\binom{n}{i}(-1)^i\sum_{j=0}^{\infty}\frac{(n-i)^j x^j}{j!},$$

which is the same as

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r = T(r,n).$$

Example 3.12. How many 3 letter words can be formed using the letters of the word "TESTBOOK"?

Proof. The generating function is $f(z) = \left(1+z+\frac{z^2}{2!}\right)^2 \cdot (1+z)^4$

The first two factors represent t and o. The other four factors represent e, x, b, k Expanding we get, $f(z) = 1 + \frac{6}{1!}z + \frac{32}{2!}z^2 + \frac{150}{3!}z^3 + \frac{606}{4!}z^4 + ...$

There can be formed 6 words with 1 letter. 32 words with 2 letters. And 150 words with 3 letters. □

4 Formulae for Reference

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \tag{49}$$

$$(1+ax)^n = \sum_{k=0}^{n}\binom{n}{k}a^n x^n \tag{50}$$

$$(1-ax)^n = \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}a^n x^n \tag{51}$$

$$(1+ax)^{-n} = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}a^k x^k \tag{52}$$

$$(1-ax)^{-n} = \sum_{k=0}^{k=\infty}\binom{n+k-1}{k}a^k x^k \tag{53}$$

$$\binom{-n}{k} = (-1)^k\binom{n+k-1}{k} \tag{54}$$

$$(x+1)^{-n} = 1 - nx + \frac{1}{2!}n(n+1)x^2 - \frac{1}{3!}n(n+1)(n+2)x^3 + \dots \tag{55}$$