Discrete Mathematics	30/01/2019
Lecture 5: Basic Counting & Inclusion-Exclusion	
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1 Introduction

This lecture note mainly concentrates on various counting problems that we come across in our daily life. Although there are no absolute rules that can be used to solve all counting problems, many counting problems that occur frequently can be solved using a few basic rules together with a few important counting techniques.
So let us go through the counting problems one by one.

2 Sampling Problem

Here by sampling problem, we mean the problem of choosing $k(\leq n)$ samples from n distinct samples. Depending upon whether we are doing ordered selection or not with repetition being allowed or not, the count of selecting k samples from n distinct samples varies.
Let us now see how it varies.

2.1 Ordered selection with repetition allowed

The process of selecting k items from a collection of n distinct objects in circumstances where the ordering of the selection matters and repetition is allowed.
In such a situation, every time we choose a sample we have n choices. That is there are n possibilities for the first choice, then there are n possibilities for the second choice, and so on.
So the Number of ways this selection can be made $= n^k$.
The notation used to denote this count is $P^R(n, k)$ (Permutation with repetition).
So, this count is equal to $= P^R(n, k) = n^k$.

2.2 Ordered selection with repetition not allowed

The process of selecting k items from a collection of n distinct objects in circumstances where the ordering of the selection matters but repetition is not allowed.
See we have to select k samples. We can select the first sample in n ways, after which there are only $(n-1)$ samples left (as repetition is not allowed). We can now select the 2^{nd} sample in $(n-1)$ ways, after which there are only $(n-2)$ samples left and so on. Finally the k^{th} sample can be selected from $(n-k+1)$samples.
So the number of way this selection is made $= n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$.
This count is nothing but $P(n, k)$.
So, this count is equal to $= P(n, k) = \dfrac{n!}{(n-k)!}$.

2.3 Unorderded selection with repetition allowed

The process of selecting k items from a collection of n distinct objects in circumstances where the ordering of the selection doesn't matter but repetition is allowed.Each selection can be represented by a list of $n-1$ bars($|$) and k stars($*$). The $n-1$ bars are used to mark off n different cells, with the i^{th} cell containing a star($*$) for each time the i^{th} element of the set occurs in the combination.

For instance, one way selection of 6 elements from a set with four elements is represented with three bars and six stars: $**|*|***|$
This represents the combination containing exactly two of the first element, one of the second element, three of the third element, and none of the fourth element of the set.

As we have seen, each different list containing $n-1$ bars and r stars corresponds to an unordered selection of r elements of the set with n elements, when repetition is allowed. The number of such lists is $\binom{n-1+r}{r}$, because list corresponds to a choice of the r positions to place the r stars from the $r+n-1$ positions that contains r stars and $n-1$ bars.
Thus the number of unordered selection of r elements from n distinct element when repetition is allowed $= C^R(n,r) = \binom{n+r-1}{r}$

2.4 Unorderded selection with repetition not allowed

The process of selecting k items from a collection of n distinct objects in circumstances where the ordering of the selection doesn't matter and repetition is also not allowed.
The easiest way to find this count is to first assume that the order matters and then divide that by the number of times the k objects can be ordered.
The number of ordered selection with no repetition $= P(n,k)$
The number of ways k objects can be ordered $= k!$
Thus The number of unordered selections with no repetition allowed $= \frac{P(n,k)}{k!}$

2.5 Summary of Sampling Problems

Does Order Matter?	**Is Repetition Allowed ?**	**Count**
Yes	Yes	$P^R(n,k) = n^k$
Yes	No	$P(n,k) = \frac{n!}{(n-k)!}$
No	Yes	$C^R(n,k) = \binom{n+k-1}{k}$
No	No	$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!}$

Table 1: SUMMARY OF SAMPLING PROBLEM

3 Counting Number Of Relations And Functions

3.1 Relations

Let R be a relation [1] on set A with n elements .
We know already that $|A \times A| = n^2$. Hence,

$$\begin{aligned} \text{Total number of Relations on set } A &= \text{Total number of subsets of } A \times A \\ &= 2^{n^2} \end{aligned}$$

Let $A = \{a_1, a_2, \cdots, a_n\}$ and we represent $A \times A$ as a matrix such that element in i^{th} row,j^{th} column represents (a_i, a_j), $1 \leq i, j \leq n$.
The matrix will be of the following form

$$\begin{bmatrix} (a_1, a_1) & (a_1, a_2) & (a_1, a_3) & \dots & (a_1, a_n) \\ (a_2, a_1) & (a_2, a_2) & (a_2, a_3) & \dots & (a_2, a_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (a_n, a_1) & (a_n, a_2) & (a_n, a_3) & \dots & (a_n, a_n) \end{bmatrix}$$

There are various types of relations.Let us see them one by one.

3.1.1 Reflexive Relations

- A relation $R \subseteq A \times A$ is *reflexive* if $(a, a) \in R \ \forall a \in A$.

Look at the matrix that represents $A \times A$ and observe that the diagonal elements of the matrix that is,$(a_i, a_i), 1 \leq i \leq n)$ are mandatory in any reflexive binary relation. Therefore, the diagonal elements of the matrix along with any subset of the remaining $n^2 - n$ elements forms a reflexive relation.
Number of subsets of a set with $n^2 - n$ elements $= 2^{n^2 - n}$.
So Number of reflexive relations $= 1 \times 2^{n^2 - n} = 2^{n(n-1)}$.

3.1.2 Symmetric Relations

- A relation $R \subseteq A \times A$ is *symmetric* if $\forall\ a, b \in A$, $(a, b) \in R \iff (b, a) \in R$.

Now, we have to find the number of such symmetric relations possible on a set A of n elements.
Consider the elements of the matrix excluding the diagonal elements. We divide them into lower triangle elements $(i > j)$ and upper triangle elements $(i < j)$. Notice that by the definition of symmetric relation, if $\exists$ an element, say $(a_i, a_j) \in R$ from the lower triangle, then the element (a_j, a_i) from the upper triangle is also in the relation R (this element is actually forced into the relation).
In total there are $\binom{n}{2} = \dfrac{n^2 - n}{2}$ such pairs of elements.Each pair has a choice to be either belong to the relation or not. Also, it is to be noted that any subset of the diagonal elements together with a subset from lower triangle (upper triangle) is a symmetric relation. Therefore,

[1] Recall that a ***Binary Relation*** on a set A is any subset R of $A \times A$.($R \subseteq A \times A$)

$$\text{The number of symmetric relations} = 2^n \times 2^{\binom{n}{2}} = 2^n \times 2^{\frac{n(n-1)}{2}} = 2^{\frac{n(n+1)}{2}}.$$

3.1.3 Antisymmetric Relations

- A relation $R \subseteq A \times A$ is *antisymmetric* if $\forall\ a, b \in A$, if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$.

Now we have to count the number of antisymmetric relations possible on a set A of cardinality n. Consider an antisymmetric binary relation and note that, if there exist an element,say $(a_i, a_j), i \neq j$ from the lower triangle, then the corresponding element (a_j, a_i) from the upper triangle should not be present in the relation and vice versa.Therefore there are 3 possibilities for an (i,j) pair.i.e., either (a_i, a_j) is in the relation, or (a_j, a_i) is in the relation, or none of $(a_i, a_j), (a_j, a_i)$ are present in the relation. There are $\binom{n}{2} = \frac{n^2 - n}{2}$ pairs of (a_i, a_j) such that $i \neq j$. Therefore, there are $3^{\binom{n}{2}} = 3^{\frac{n^2-n}{2}}$ relations. Also, observe that any subset of the diagonal elements are possible in an antisymmetric relation. Therefore,

$$\text{The number of antisymmetric binary relations} = 2^n \times 3^{\binom{n}{2}} \& = 2^n \times 3^{\left(\frac{n(n-1)}{2}\right)}$$

3.1.4 Transitive Relations

- A binary relation $R \subseteq A \times A$ is *transitive* if, $\forall\ a, b, c \in A$, if $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R$.

Before counting the number of transitive relations that can be defined on a set A with n elements, let us define the following terms:

1. **Partial Order:** A *partial order* on a set A is a binary relation on A which is transitive, reflexive and anti-symmetric.
2. **Augmented Partition:** We call a set X to be an *augmented partition* of A if X is the union of a subset $Y \subseteq A$ and a partition L of its complement $A \setminus Y$.i.e. $X = Y \cup L$

Suppose we are given a transitive relation R on the set A with n elements.

Claim: *Every transitive relation R on A is an augmented partition $L \cup Y$ of A together with a partial order on $L \cup Y$, and vice-versa.*

Proof:

$(\implies)$
Suppose we are given a transitive relation R on A.
Define $Y = \{a \in A | (a,a) \notin R\}$.

Consider the *symmetric core* $\equiv$ defined as: $a \equiv b \iff (a, b) \in R$ and $(b, a) \in R$.

Clearly,

- If $a \in A \setminus Y$, then $a \equiv a$ (by transitivity of R).
- If $a, b \in A \setminus Y$, then $a \equiv b \iff b \equiv a$ (by definition of $\equiv$).
- If $a, b, c \in A \setminus Y$ such that $a \equiv b$ and $b \equiv c$,
  then
  $(a.b) \in R$, $(b, c) \in R \implies (a, c) \in R$
  and
  $(c, b) \in R$, $(b, a) \in R \implies (c, a) \in R$,
  hence $a \equiv c$.

Thus $\equiv$ **is an *equivalence relation* on** $A \setminus Y$**.**

This equivalence relations partitions $A \setminus Y$ into equivalence classes. Call this partition $L := (A \setminus Y)/\equiv$.
Consider the augmented partition X of A, where $X = L \cup Y$. Define the elements of X by $[x]$, where $[x]$ denotes $\equiv$-equivalence class of x if $x \in A \setminus Y$, and the element x if $x \in Y$.

Define a relation $\sqsupset$ on X, such that $([x], [y]) \in \sqsupset \iff (x, y) \in R$.

Note that $\sqsupset$ is well defined:

- Let $x, y \in A \setminus Y$ such that $u \in [x]$, $v \in [y]$ such that $u \neq x$, $v \neq y$, and let $([x], [y]) \in \sqsupset$.
  Then
  $u \equiv x \implies (u, x) \in R$, and $(x, u) \in R$,
  and
  $v \equiv y \implies (v, y) \in R$, and $(y, v) \in R$.
  And also by definition of $\sqsupset$, $(x, y) \in R$.
  This implies, by transitivity of R, that, $(u, v) \in R$.

- If $x \in A \setminus Y$ such that $u \in [x]$, $u \neq x$, and $y \in Y$,
  then
  $(u, x) \in R$,
  and
  $(x, y) \in R \implies (u, y) \in R$.

  The case when $x, y \in Y$, is, trivial.

Now,

- $\sqsupset$ is *transitive* by transitivity of R .
- If $x \in A \setminus Y$ such that $x, y \in [x]$,
  then $x \equiv y$,
  and hence $(x, y) \in R$.
  Hence $([x], [y]) \in \sqsupset$,
  and since $y \in [x]$, hence $[y] = [x]$, so that $([x], [x]) \in \sqsupset$,
  So, $\sqsupset$ is *reflexive*.

- If $([x],[y]) \in \sqsupset$, and $([y],[x]) \in \sqsupset$
  then
  $(x,y) \in R$ and $(y,x) \in R$,
  Which implies
  $x \equiv y \implies [x] = [y]$,
  i.e $\sqsupset$ is *anti-symmetric*.

  Thus $\sqsupset$ defines a ***partial order*** on $X = L \cup Y$.

$(\Longleftarrow)$
Given an augmented partition $X = L \cup Y$ of A with a partial order $\sqsupset$ on X, define R on A by $(x,y) \in R \iff ([x],[y]) \in \sqsupset$.

Define $\equiv$ on $A \setminus Y$ by $x \equiv y \iff x$ and y belong to same partition set. Thus clearly $\equiv$ is an equivalence relation on $A \setminus Y$.

Denote by $[x]$ the partition set containing x. R is well-defined (by the similar logic as the previous one). Note that R is *transitive* by the transitivity of $\sqsupset$. $\square$

Thus this claim means, the number of transitive relations on A is equal to the number of augmented partitions $X = L \cup Y$ of A with a partial order on X.

Let $T(A)$ be the number of transitive relations o A, $|A| = n$.
If $P(X)$ denote the number of partial order on set X, then:

$$T(A) = T(n) = \sum_{k=0}^{n} \sum_{s=0}^{k} \sum_{Y} \sum_{L} P(L \cup Y) \cdots\cdots (1)$$

where the sum is over all $Y \subseteq X$ with $|Y| = s$, and all partitions L of $A \setminus Y$ into $k-s$ non-empty parts.

Hence $|L \cup Y| = k$, so we can write $P(L \cup Y) = P(k)$, where $P(k)$ denotes the number of partial orders that can be defined on a set with cardinality k .

Also, number of possible choices of Y is $\binom{n}{s}$ and for each such choice, there are $S(n-s, k-s)$ partitions of $A \setminus Y$ into $k - s$ non-empty parts, where $S(n,i)$ is *Stirling's number of second kind.*

Hence putting these in (1),

$$T(n) = \sum_{k=0}^{n} \left(\sum_{s=0}^{k} \binom{n}{s} S(n-s, k-s) \right) P(k)$$

3.1.5 Equivalence Relations

- A binary relation $R \subseteq A \times A$ is an *equivalence relation* $\iff$ R is reflexive, symmetric and transitive. That is, $\forall\, a,b,c \in A$:
  1. $(a,a) \in R$ (reflexivity)
  2. $(a,b) \in R \iff (b,a) \in R$ (symmetry)
  3. If $(a,b) \in R$,$(b,c) \in R$, then $(a,c) \in R$ (transitivity).

Let R be an equivalence relation on A. We partition A as follows: if a,$b \in A$, and $(a,b) \in R$ then put a and b in the same partition. Thus an equivalence relation on A effectively implies a partition of the set A. Conversely, given any partition of set A, we can define a relation R so that $(a,b) \in R \iff a$

and b belong to the same block in that partition. Clearly this is an equivalence relation. Thus we can see that number of equivalence relations on set A is same as the number of partitions of set A.
We know that the number of partitions of a set of cardinality n into i classes= $S(n,i)$
where $S(n,i) = \frac{1}{i!}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}(k)^n$
and $S(n,i)$ is known as *Stirling's number of second kind.*
Hence the number of partitions of set A (with cardinality n) $B_n = \sum_{i=0}^{n} S(n,i)$. This number is known as *Bell's Number.*

3.1.6 Relations which are both reflexive and symmetric Relations

By now, we are well acquainted with both reflexive relations and symmetric relations.
We also know that all diagonal elements $(a_i, a_i), 1 \le i \le n$ are part of the reflexive relation and there exists 2 possibilities for the remaining $\binom{n}{2} = \frac{n(n-1)}{2}$ elements.Each $(a_i, a_j), (a_j, a_i)$pair has a choice to be either belong to the relation or not. Thus,

$$\begin{aligned}\text{The total number of relations that are both reflexive and symmetric } &= 2^{\binom{n}{2}}\\ &= 2^{\left(\frac{n(n-1)}{2}\right)}\end{aligned}$$

3.1.7 Relations which are both reflexive and antisymmetric

We already know what reflexive and antisymmetric relations are.
We also know that all diagonal elements $(a_i, a_i), 1 \le i \le n$ are part of the reflexive relation and there exists 3 possibilities for the remaining $\binom{n}{2} = \frac{n(n-1)}{2}$ elements .i.e., either (a_i, a_j) is in the relation, or (a_j, a_i) is in the relation, or none of $(a_i, a_j), (a_j, a_i)$ are present in the relation
Hence,

$$\begin{aligned}\text{The total number of relations that are reflexive and antisymmetric} &= 3^{\binom{n}{2}}\\ &= 3^{\left(\frac{n(n-1)}{2}\right)}\end{aligned}$$

3.1.8 Summary of Relations

TYPE OF RELATIONS	NUMBER OF RELATIONS
all relations	2^{n^2}
reflexive	$2^{n(n-1)}$
symmetric	$2^{\frac{n(n+1)}{2}}$
equivalence	$B_n = \sum_{i=0}^{n} S(n,i)$
antisymmetric	$2^n \times 3^{\frac{n(n-1)}{2}}$
transitive	$\sum_{k=0}^{n} \left(\sum_{s=0}^{k} \binom{n}{s} S(n-s, k-s) \right) P(k)$
reflexive and symmetric	$2^{\binom{n}{2}}$
reflexive and antisymmetric	$3^{\binom{n}{2}}$

Table 2: Summary of Relations

.
We know already that $|A \times A| = n^2$.

3.2 Functions

$f : A \to B$ is a function [2]

Let us now find the total number of functions from A to B.
Let $|A| = m$; $|B| = n$.
We have to map m elements of A to n elements in B. Note that the order of mapping matters and repetition is allowed. So it is just an ordered selection of m elements from n elements of B when repetition is allowed. (Refer the counting problem we mentioned in 2.1.)
Hence,

$$\text{The Number of functions from } A \text{ to } B = n^m$$

3.2.1 Injective Functions

- A function $f : A \to B$ is *injective* if $f(x) = f(y) \implies x = y$.
  In other words different elements of A have different images in B.

Now we have to find number of injective functions from A to B.
Let $|A| = m; |B| = n$. We have to map m elements of A to m elements in B such that no two elements

[2]Recall that a ***function*** $f : A \to B$ is a relation between A and B (i.e. $f \subseteq A \times B$) such that $\forall x \in A$ $\exists\,!\, y \in B$ such that $(x, y) \in f$.

in A are mapped to the same element. Note that the order of mapping matters and repetition is not allowed.So it is just an ordered selection of m elements from n elements of B when repetition is not allowed.(Refer the counting problem which we have done in 2.2.) Thus,

$$\text{Number of Injective functions} = P(n, m)$$

3.2.2 Surjective Functions

- A function $f : A \leftarrow B$ is *surjective* if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$. (Sometimes also called onto.)

To count the number of surjective functions we have to use the theorem 4.1. Let A and B be finite sets, and $|A| = m, |B| = n$. Counting the functions of the form $f : A \to B$ is easy if you know the principle of inclusion-exclusion. Each $x \in A$ has n choices for its image, the choices are independent, and therefore the number of functions is n^m (as we have already seen above).
How many of these functions are surjective ?
To answer this question, let $B = \{y_1, y_2, ..., y_n\}$ and let A_i be the set of functions in which y_i is is not the image of any element in A.
So here the property corresponding to set A_i is defined as $P_i := y_i$ is left out.

$$\begin{aligned}
\text{Number of surjective functions} &= N(P'_1 P'_2 \cdots P'_n) \\
&= N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) - \cdots + (-1)^n N(P_1 P_2 ... P_n) \\
&= n^m - n(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}(n-(n-1))^m + \\
&(-1)^n \binom{n}{n}(n-(n))^m \\
&= \sum_{i=0}^{n} \binom{n}{i}(-1)^i (n-i)^m
\end{aligned}$$

Let $S =$ Number of surjective functions from A to B

$$S = \begin{cases} 0, & \text{for } m < n \\ n!, & \text{for } m = n \\ \sum_{i=0}^{n} \binom{n}{i}(-1)^i (n-i)^m & \text{for } m > n \end{cases}$$

3.2.3 Bijective Functions

- function f is *bijective* if it is both injective and surjective.

For a function $f : A \to B$ to be bijective $|A|$ must be equal to $|B|$.
Let $|A| = |B| = n$.
We have to map n elements of A to n elements of B in an orderly way when no repetition allowed.(Refer 2.2)
Hence,

$$\begin{aligned}
\text{Number of bijective functions} &= P(n, n) \\
&= \frac{n!}{(n-n)!} \\
&= n!
\end{aligned}$$

3.2.4 Summary of functions

Let $|A| = m, |B| = n$;

TYPE OF FUNCTIONS $f : A \to B$	**NUMBER OF FUNCTIONS**
all functions	n^m
injective	$P(n, m)$
surjective	$S = \begin{cases} 0, & \text{for } m < n \\ n!, & \text{for } m = n \\ \sum_{i=0}^{n} \binom{n}{i}(-1)^i(n-i)^m & \text{for } m > n \end{cases}$
bijective	$n!$

Table 3: Summary of Functions

4 Principle of Inclusion-Exclusion

Consider a set S with N elements. Let A_i be the subset of elements in a set S ($A_i \subseteq S$) that have property P_i. Let us denote the number of elements of S that have properties P_i by $N(P_i)$. Observe that $N(P_i) = |A_i|$ as elements in S that shows propertyP_i forms set A_i. Now consider r properties i.e $P_1, \cdots .., P_r$ The number of elements that have properties $P_1, P_2, ..., P_r$ is often written $N(P_1P_2...P_r)$ (it is same as$\left|A_1 \cap A_2 \cap \cdots . \cap A_r\right|$) and the number of elements that have none of these properties is often written $N(P_1'P_2' \cdots P_r')$.(it is equal to $\left|A_1' \cap A_2' \cap \cdots . \cap A_r'\right|$).

Theorem 4.1 (PRINCIPLE OF INCLUSION-EXCLUSION [3])**.** *Let S be a finite set of N elements and $A_1, A_2, \cdots .A_r$ be subsets of S with elements having properties $P_1, P_2, ..., P_r$ respectively. Then the number of elements having none of properties $P_1, \cdots .., P_r$ is given by ;*

$$N(P_1'P_2' \cdots P_r') = N - \sum_{1 \le i \le r} N(P_i) + \sum_{1 \le i < j \le r} N(P_iP_j) - \sum_{1 \le i < j < k \le r} N(P_iP_jP_k) + \cdots\cdots . + (-1)^r N(P_1P_2...P_r) \tag{1}$$

We give two proofs of this theorem.

[3]Recall that we have learned in semester 1 the following form of inclusion-exclusion principle.
The principle of inclusion and exclusion states that for finite n sets $A_1, A_2, \cdots .A_n$ the following holds.

$$|\bigcup_{1 \le i \le n} A_i| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \cdots\cdots . + (-1)^{(n+1)} |\bigcap_{1 \le i \le n} A_i|$$

4.1 Proof of Theorem 4.1

4.1.1 Proof by induction:

We proceed via *induction on* r. For $r = 1$;

$$LHS = N(P_1');$$
$$RHS = N - N(P(_1) = N(P_1') = LHS;$$

So the theorem holds for $r = 1$;
For the induction step, let us suppose the theorem holds for $r - 1$.
Equation (1) gives us;

$$N(P_1'P_2'\cdots P_{r-1}') = N - \sum_{1\le i\le(r-1)} N(P_i) + \sum_{1\le i<j\le(r-1)} N(P_iP_j) - \cdots\cdots . + (-1)^{(r-1)} N(P_1P_2...P_(r-1)) \tag{2}$$

We have to prove that theorem holds for r. i.e;

$$N(P_1'P_2'\cdots P_r') = N - \sum_{1\le i\le r} N(P_i) + \sum_{1\le i<j\le r} N(P_iP_j) - \sum_{1\le i<j<k\le r} N(P_iP_jP_k) + \cdots\cdots . + (-1)^r N(P_1P_2...P_r)$$

Now, we note that:

$$\begin{aligned} LHS &= N(P_1'P_2'\cdots P_r') \\ &= \left|A_1' \cap A_2' \cap \cdots . \cap A_r'\right| \end{aligned}$$

Now we use the formula $|X \cap Y| = |X| + |Y| - |X \cup Y|$ and we get;

$$\begin{aligned} LHS &= \left|(A_1' \cap A_2' \cap \cdots . \cap A_{r-1}') \cap A_r'\right| \\ &= \left|(A_1' \cap A_2' \cap \cdots . \cap A_{r-1}')\right| + \left|A_r'\right| - \left|(A_1' \cap A_2' \cap \cdots . \cap A_{r-1}') \cup A_r'\right| \\ &= \left|(A_1' \cap A_2' \cap \cdots . \cap A_{r-1}')\right| + \left|A_r'\right| - \left|(\bigcap_{i=1}^{r-1} A_i') \cup A_r'\right| \\ &= \left|(\bigcap_{i=1}^{r-1} A_i')\right| + \left|A_r'\right| - \left|(\bigcap_{i=1}^{r-1} A_i' \cup A_r')\right| \end{aligned}$$

Let us define $B_i = A_i \cap A_r \implies B_i' = A_i' \cup A_r'$
Assume $Q_i(= P_iP_r)$ is the property corresponding to set B_i.
We can now use the induction hypothesis (2) on both $\bigcap_{i=1}^{r-1} A_i'$ and $\bigcap_{i=1}^{r-1} B_i'$.

LHS
$= |\bigcap_{i=1}^{r-1} A_i'| + |A_r'| - |\bigcap_{i=1}^{r-1} B_i'|$
$= N(P_1'P_2'\cdots P_{r-1}') + N(P_r') - N(Q_1'Q_2'\cdots Q_{r-1}')$

$$= \left[N - \sum_{1 \le i \le (r-1)} N(P_i) + \sum_{1 \le i < j \le (r-1)} N(P_iP_j) - \cdots + (-1)^{(r-1)} N(P_1P_2..P_{(r-1)})\right] + [N - N(P_r)]$$
$$- \left[N - \sum_{1 \le i \le (r-1)} N(Q_i) + \sum_{1 \le i < j \le (r-1)} N(Q_iQ_j) - \cdots + (-1)^{(r-1)} N(Q_1Q_2...Q_{(r-1)})\right]$$

$$= \left[N - \sum_{1 \le i \le (r-1)} N(P_i) + \sum_{1 \le i < j \le (r-1)} N(P_iP_j) - \cdots + (-1)^{(r-1)} N(P_1P_2..P_{(r-1)})\right] + [N - N(P_r)]$$
$$- \left[N - \sum_{1 \le i \le (r-1)} N(P_iP_r) + \sum_{1 \le i < j \le (r-1)} N(P_iP_jP_r) - \cdots + (-1)^{(r-1)} N(P_1P_2...P_{(r-1)}P_r)\right]$$

$$= N - \left[\sum_{1 \le i \le (r-1)} N(P_i) - N(P_r)\right] + \left[\sum_{1 \le i < j \le (r-1)} N(P_iP_j) + \sum_{1 \le i \le (r-1)} N(P_iP_r)\right] - \cdots + (-1)^r N(P_1P_2 \cdots P_r)$$

$$= N - \sum_{1 \le i \le r} N(P_i) + \sum_{1 \le i < j \le r} N(P_iP_j) - \sum_{1 \le i < j < k \le r} N(P_iP_jP_k) + \cdots\cdots . + (-1)^r N(P_1P_2...P_r)$$
$$= N(P_1'P_2' \cdots P_r')$$

=RHS

Hence we have proved that the theorem holds for 'r' if it holds for '$r-1$'.
Thus by *principle of mathematical induction* we have proved Theorem 4.1. □

4.1.2 Proof by counting:

The key idea is to think of the the elements of set S individually and ask what each one contributes to the sum in Eqn. (1). Take an arbitrary element $x \in S$.
There are two possibilities for $x \in S$. They are;

1. x doesn't have any of the r properties $(P_1, \cdots ..P_r)$.
2. x has exactly p out of the r properties $(P_1, \cdots ..P_r)$.

Case 1: Suppose $x \in S$ doesn't have any of the r properties $(P_1, \cdots ..P_r)$.

So contribution of x to $N(P_1'P_2' \cdots P_r')$ is 1 in L.H.S.of Eqn. 4.1

Claim: In R.H.S of Eqn. (1) x is counted exactly once.
Proof: Eqn. (1) is:

$$N(P_1'P_2' \cdots P_r') = N - \sum_{1 \le i \le r} N(P_i) + \sum_{1 \le i < j \le r} N(P_iP_j) - \sum_{1 \le i < j < k \le r} N(P_iP_jP_k) + \cdots + (-1)^r N(P_1P_2 \cdots P_r)$$

Note that the x is counted exactly once in N. The rest of terms in R.H.S. counts elements showing the properties $(P_1, \cdots ..P_r)$. Since x doesn't show any of the properties it is counted 0 times in the other terms in R.H.S. excluding N. So in total x is counted exactly once in R.H.S.
So the count of x in L.H.S = count of x in L.H.S=1;

Case 2: Suppose x has exactly p out of the r properties $(P_1, \cdots P_r)$ where $1 \le p \le r$.

In L.H.S x is counted 0 times .

Claim: In R.H.S Eqn. (1) x is counted 0 times.
Proof:
• x is counted exactly once in N(1^{st} term of R.H.S).
• x is counted p times in $\sum_{1\leq i\leq r} N(P_i)$ (2^{nd} term of R.H.S) as x shows exactly p out of the r properties and for other properties the contribution of x to the count is 0.
• x is counted $\binom{p}{2}$ times in $\sum_{1\leq i<j\leq r} N(P_iP_j)$ (3^{rd} term of R.H.S)
• So for the l^{th} term such that $1 \leq l \leq p+1$,x is counted exactly $\binom{p}{l-1}$ times.
• For all other terms contribution of x to the count is 0.
So in R.H.S the number of times x is counted $= 1 - p + \binom{p}{2} - \binom{p}{3} + \cdots + (-1)^p\binom{p}{p} = (1-1)^p = 0$
Hence Claim Proved.
So in L.H.S as well as R.H.S the count of $x = 0$. □

5 General Principle of Inclusion-Exclusion

We can generalize the Inclusion-Exclusion theorem to include exactly m of the properties in our count. Before doing that, we define some notations for ease of computation:

$$s_0 = N$$
$$s_1 = \sum_{i=1}^{r} N(P_i)$$
$$s_2 = \sum_{i<j}^{r} N(P_iP_j)$$
$$s_3 = \sum_{i<j<k}^{r} N(P_iP_jP_k)$$
$$\vdots$$
$$s_r = N(P_1P_2P_3\cdots P_r)$$

where $N(P_{i_1}P_{i_2}\cdots a_{i_k})$ has been defined in section 4.
Let
e_0= number of objects that have 0 of the properties.
e_1= number of objects that have exactly 1 of the properties.
And in general, e_m=number of objects that have exactly m of the properties.

Now we state our generalized version:

Theorem 5.1 (GENERAL PRINCIPLE OF INCLUSION-EXCLUSION)**.** *Let S be a finite set of N elements and $A_1, A_2, \cdots .A_r$ be subsets of S with elements having properties $P_1, P_2, ..., P_r$ respectively.Then the number of elements having exactly m of the properties $P_1, \cdots .., P_r$ is given by ;*

$$e_m = s_m - \binom{m+1}{1} s_{m+1} + \binom{m+2}{2} s_{m+2} + \cdots + (-1)^{r-m} \binom{r}{r-m} s_r \tag{3}$$

$$= \sum_{k=0}^{r-m} (-1)^k \binom{m+k}{k} s_{m+k} \tag{4}$$

Note that in Theorem 4.1 we calculated e_0.

5.1 Proof of Theorem 5.1

Take an arbitrary element $x \in S$.
There are three possibilities for $x \in S$. They are;

1. x has less than m properties out of the r properties $(P_1, \cdots ..P_r)$.

2. x has exactly m properties out of the r properties $(P_1, \cdots ..P_r)$.

3. x has more than m properties out of the r properties $(P_1, \cdots ..P_r)$.

Case 1: Suppose $x \in S$ has less than m properties out of the r properties $(P_1, \cdots ..P_r)$.

So contribution of x in L.H.S. of equation (3) to e_m is 0
Now let us look at R.H.S .Every term of R.H.S contains s_k s.t $k \geq m$. Recall that we defined s_k such that it counts the number of elements showing k properties. Since x has less than m properties, it is not counted in any of the s_k $(k \geq m)$.
The count of x in R.H.S of equation (3) is also 0

Case 2: Suppose $x \in S$ has exactly m properties out of the r properties $(P_1, \cdots ..P_r)$.

In L.H.S of equation (3) x is counted exactly once as we are counting elements with exactly m properties.
In R.H.S of equation (3), x is counted once in the term s_m .In the remaining terms,x contributes 0 to the count as these terms contains s_{m+j} where $1 \leq j \leq r$.
So in the R.H.S of equation (3) also, x is counted exactly once.

Case 3: Suppose $x \in S$ has more than m properties out of the r properties $(P_1, \cdots ..P_r)$.

In L.H.S of equation (3) x is counted 0 times as it has more than m properties and we are only counting elements with exactly m properties.
Claim: In R.H.S equation (3) x is counted 0 times.
Proof:
Suppose x has $m+j$ properties, then for $0 \leq k \leq j$, x is counted $\binom{m+j}{m+k}$ times in s_{m+k}.

So the total number of times x is counted in R.H.S of equation (3) is given by

$$\begin{aligned}\sum_{k=0}^{j}(-1)^k\binom{m+k}{k}\binom{m+j}{m+k} &= \sum_{k=0}^{j}(-1)^k\binom{m+j}{j}\binom{j}{k} \\ &= \binom{m+j}{j}\sum_{k=0}^{j}(-1)^k\binom{j}{k} \\ &= \binom{m+j}{j}\times(1-1)^j \\ &= 0\end{aligned}$$

Here we used the identity

$$\begin{aligned}\binom{m+k}{k}\binom{m+j}{m+k} &= \frac{(m+k)!}{m!k!}.\frac{(m+j)!}{(j-k)!(m+k)!} \\ &= \frac{(m+j)!}{m!k!(j-k)!} \\ &= \frac{(m+j)!}{m!k!(j-k)!}\frac{j!}{j!} \\ &= \frac{(m+j)!}{m!j!}.\frac{j!}{k!(j-k)!} \\ &= \binom{m+j}{j}\binom{j}{k}\end{aligned}$$

□

5.2 Applications of Principle of inclusion-exclusion

1. ***Finding Number of Surjective Functions***
   We have already found out the number of surjective functions using principle of inclusion-exclusion(Theorem 4.1).

$$S = \begin{cases} 0, & \text{for } m < n \\ n!, & \text{for } m = n \\ \sum_{i=0}^{n}\binom{n}{i}(-1)^i(n-m)^m & \text{for } m > n \end{cases}$$

   where S= Number of surjective functions.
   For more refer section(3.2.2)

2. ***Dearangements***
   A permutation of $1, 2, \cdots n$ is a derangement if i is not in the i^{th} position for all $i = 1, 2, \cdots, n$.
   We want to count the number of derangements, d_n on n objects.
   To use the principle of inclusion $-$ exclusion, we define our objects and properties:
   The objects are all the permutations of $\{1, 2, \cdots, n\}$. Property P_i is i is in the i^{th} position.
   $d_n = e_0$ then.

$$
\begin{aligned}
N(P_i) &= (n-1)! \\
N(P_iP_j) &= (n-2)! \\
N(P_iP_jP_k) &= (n-3)! \\
s_0 &= n! \\
s_1 &= n.(n-1)! \\
s_2 &= \binom{n}{2}.(n-2)! \\
s_2 &= \binom{n}{2}.(n-3)! \\
&\vdots \\
s_k &= \binom{n}{k}.(n-k)!
\end{aligned}
$$

Then the number of dearangements is given by

$$
\begin{aligned}
d_n &= e_0 \\
&= s_0 - s_1 + s_2 + \cdots + (-1)^r s_r \\
&= n! - n.(n-1)! + \binom{n}{2}.(n-2)! - \binom{n}{2}.(n-3)! \cdots (-1)^n.\binom{n}{n}.(n-n)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \\
&\sim n! e^{-1}
\end{aligned}
$$

□

6 Occupancy Problem

6.1 r distinct balls into n distinct boxes, empty cells are allowed

In this case, we consider putting r balls, numbered 1 through r, into n boxes, numbered 1 through n, but this time with no restriction on the number of balls that can go into each box. Instead of thinking in terms of putting r balls into n boxes, we can think in terms of selecting r of the n boxes.In this case more than one ball may go into the same box, which means that the same box may be chosen more than once.
Here we are dealing with ordered selections, of the boxes, with repetition allowed (Refer section 2.1)
Therefore, Number of ways to distribute r distinguishable balls into n distinguishable boxes when empty boxes are allowed=$P^R(n, r) = n^r$

6.2 r distinct balls into n distinct boxes, empty cells are not allowed

The number of ways to distribute $r(\geq n)$ distinguishable balls into n distinguishable boxes, in such a way that no box is empty can be found out using principle of inclusion-exclusion (Refer 4.1). Let

C_i be the property that the i^{th} box is empty and $1 \leq i \leq n$.
The method is same as finding number of surjective functions from $f : \{1, \cdot sr\} \to \{1, 2, \cdot sn\}$.So

$$
\begin{aligned}
\text{Number of ways} &= N(C_1'C_2'\cdots C_n') \\
&= N - \sum_{1\leq i\leq n} N(C_i) + \sum_{1\leq i<j\leq n} N(C_iC_j) - \cdots + (-1)^n N(C_1C_2...C_n) \\
&= n^r - n(n-1)^r + \binom{n}{2}(n-2)^r - \cdots + (-1)^n \binom{n}{n-1}(n-(n))^r \\
&= \sum_{i=0}^{r} \binom{n}{i}(-1)^i (n-r)^r
\end{aligned}
$$

The number of ways to distribute $r(\geq n)$ distinguishable balls into n distinguishable boxes, in such a way that no box is empty $= \sum_{i=0}^{n} \binom{n}{i}(-1)^i(n-r)^r = n! \times \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i(n-r)^r = n! \times S(r,n) = T(r,n)$
Here $S(r,n) = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i(n-r)^r$ and is called *Stirling number of the second kind.*

6.3 r distinct balls into n identical boxes, empty cells are allowed

We firstly consider how many empty cells are there. Suppose, for example, there is only one empty cell. So the distribution of r distinct balls into n identical boxes, such that there is only 1 empty cell, is *equivalent* to the r distinct balls into $n-1$ identical boxes, such that there is no empty cell, which by subsection 6.4, is, $S(r, n-1)$. In general, if there are exactly k empty boxes, $k = 0, \cdots, n-1$, we'll have $S(r, n-k)$ no. of ways of distribution.
Thus, total no. of ways of distributing r distinct balls into n identical boxes, empty cells are allowed $= \sum_{k=0}^{n-1} S(r, n-k) = \sum_{k=1}^{n} S(r,k) = B(r,n)$.

6.4 r distinct balls into n identical boxes, empty cells are not allowed

Let the r distinct balls correspond to the integers $1, \cdots, r$. The distribution of r distinct balls to n identical boxes corresponds to the the number of partitions of a set of r elements into n blocks. Now, if we designate the n identical boxes as $1, \cdots, n$, we'll get the exact same count as that of 6.2. But since, we have identical boxes, by designating the boxes we have actually counted each arrangement of balls $n!$(= the number of permutations of the n designated boxes) times. So to get the count for identical boxes, we have to count out this $n!$ permutations for each arrangement of r distinct balls, i.e we need to divide by $n!$.
Thus, the number of ways to distribute r distinct balls into n identical boxes, is:

$$\frac{T(r,n)}{n!} = S(r,n)$$

where $S(r,n)$ is *Stirling's number of Second Kind.*

6.5 r identical balls into n distinct boxes, empty cells are allowed

In this case, we have r identical balls, to be distributed into n distinguishable boxes, but with no restriction on the number of balls that can occupy a given box. Since the balls are indistinguishable, we can only tell how many balls each box has received. This translates into making a choice of r

of the n boxes, but with the possibility that a box may be chosen more than once. Thus, placing r balls into n boxes in this case corresponds to forming an unordered selection, or combination, of size r, taken from the set of n boxes, but with unrestricted repetitions (Refer section 2.3).
So number of ways to distribute r indistinguishable balls into n distinguishable boxes when empty cells are allowed $= C^R(n,r) = \binom{n+r-1}{r}$.

6.6 r identical balls into n distinct boxes, empty cells are not allowed

We have to distribute r identical balls into n distinct boxes,with no box being empty. For this case $r \geq n$.Let us first fill all n boxes with one ball each (such that no box is empty).So our problem reduces to distribution of remaining $(r-n)$ identical balls into n distinction boxes, when empty cells are allowed. From section 6.5 we know that number of ways this can be done $= C^R(n, r-n)$ $= \binom{r-1}{r-n}$.
So number of ways to distribute r indistinguishable balls into n distinguishable boxes when empty cells are not allowed $= C^R(n, r-n) = \binom{r-1}{r-n}$

6.7 Summary

ARE BALLS (B) DISTINGUISH-ABLE?	ARE CELLS (C) DISTINGUISH-ABLE?	ARE EMPTY CELLS (C) AL-LOWED?	COUNT
Y	Y	Y	$P^R(n,r) = n^r$
Y	Y	N	$T(r,n) = n!S(r,n)$
Y	N	Y	$\sum_{i=1}^{n} S(r,i)$
Y	N	N	$S(r,n) = \frac{T(r,n)}{n!}$
N	Y	Y	$C^R(n,r)$
N	Y	N	$C^R(n,r-n)$
N	N	Y	$\pi_1^r + \pi_2^r + \ldots + \pi_n^r$.
N	N	N	π_n^r

Table 4: Summary of Occupancy Problems

Here, π_p^r denote the number of partitions of a positive integer r into exactly p $(\leq r)$ parts.