Discrete Mathematics	09/01/2019
Lecture 1: Sets, Relations, Functions	
Instructor: Goutam Paul	*Scribe: Aditya Guha Roy*

1 Sets

The most common definition of a set we've seen till now is as follows :

Definition 1.1. *A set is a well defined collection of distinct objects.*

Examples include "a *pack* of *wolves*" , "a *deck* of *cards*", etc..
Observe that a salient feature illustrated by these examples (and every other example we may have seen so far) is that it speaks about the entities it contains (viz., wolves, cards, etc.). This motivates us towards the following :

Definition 1.2. *An object present in a set is known as an element of the set.*

Observe that the definition (we presented above) introduces three words

- collection
- objects
- well defined (collections)

We already have some idea about the terms "collection" and "objects". But the term "well defined (collection)"[1] isn't what we use very frequently. Indeed this motivates us towards asking : **Question** When do we call a collection well defined ?

And the answer to it is : we shall call a given collection $\mathcal{C}$ well defined if given any arbitrary object θ, we can answer whether θ is an element of the collection $\mathcal{C}$ or not. **Example** $\{1, 2, a, b, 2x^{2019} - 2019x^2, \star, \{2, 0, 1, 9\}\}$ is a set. Now that we know about sets, we can ask questions about whether something belongs[2] to a set or not.
So as per what we said till now, asking whether a set contains itself or not is a valid question. A related question asks whether the set R that contains all and only those x which has the property $x \notin x$ contains itself or not. And this is rather a paradox, because $R \in R \implies R \notin R$ and $R \notin R \implies R \in R$. This is popularly known as the Russel paradox named after Bertrand Russel who discovered the paradox in 1901.

This points at some thorny issues our naive treatment of sets can lead us to. Thus we realize a need to formalize our treatment of sets and this is what is done in Axiomatic set theory.

Broadly the axiomatic study of sets can be classified into two categories :

[1] note that we are not focusing up on the term or meaning of the term "well defined". We are instead more focused up on the term "well defined collection" – that is what do we mean by saying a collection is well defined.

[2] we say an object x belongs to a set χ to mean that x is an element of χ and we write $x \in \chi$ to denote the same.

- The Zermelo Fraenkel set theory without the Axiom of Choice
- The Zermelo Fraenkel set theory with the Axiom of Choice

While the former allows a room for asking questions like $x \overset{?}{\in} x$ for certain x, the latter doesn't do so.

But we'll not discuss these things at present. Before we terminate this section, we'll mention the Axiom of Choice.
The Axiom of Choice says that a non-empty Cartesian product[3] of non-empty sets is non-empty. [4]

1.1 Set operations

Definition 1.3. *A set $\mathcal{A}$ is called a* ***subset*** *of a set $\mathcal{B}$ if each element of $\mathcal{A}$ is also a member of $\mathcal{B}$. And we write $\mathcal{A} \subseteq \mathcal{B}$ to denote the same.*

A related concept is that of proper subset (a.k.a. strict subset).

Definition 1.4. *We would call a set A to be a* ***proper subset*** *of B if each element of A is also an element of B but there is at least one member in B which is not an element of A. And we write $A \subset B$ to denote this*[5]*.*

We can define equality of sets as follows :

Definition 1.5. *For two sets $\mathcal{A}, \mathcal{B}$ we say $\mathcal{A} = \mathcal{B}$ if they contain exactly the same elements.*

And using the definition of subsets (as above) this definition of equality of sets translates to $\mathcal{A} = \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.

Definition 1.6. *For sets $\mathcal{A}, \mathcal{B}$ we define the set $\mathcal{A} \cup \mathcal{B}$ to be the set of all and only those objects which are member of at least one of the sets $\mathcal{A}, \mathcal{B}$. This set $\mathcal{A} \cup \mathcal{B}$ is called $\mathcal{A}$* ***union*** *$\mathcal{B}$.*

Definition 1.7. *For sets $\mathcal{A}, \mathcal{B}$ we define the set $\mathcal{A} \cap \mathcal{B}$ to be the set of all and only those objects which are member of both the sets $\mathcal{A}, \mathcal{B}$. This set $\mathcal{A} \cap \mathcal{B}$ is called $\mathcal{A}$* ***intersection*** *$\mathcal{B}$.*

Definition 1.8. *For sets $\mathcal{A}, \mathcal{B}$ we define the set $\mathcal{A} \setminus \mathcal{B}$ to be the set of all and only those objects which are member of the set $\mathcal{A}$ but are not members of the set $\mathcal{B}$.*

Definition 1.9. *For sets $\mathcal{A}, \mathcal{B}$ we define the set $\mathcal{A} \triangle \mathcal{B}$ to be the set* $(\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A})$

[3] a non-empty Cartesian product means that we are taking Cartesian product of the elements of a non-empty set of non-empty sets

[4] and from the Axiom of Choice along with some other axioms of the Zermelo Fraenkel set theory we can explicitly prove that $x \overset{?}{\in} x$ doesn't make a valid question.

[5] often the alternative notation $\subsetneq$ is also used for the same purpose.

The following exercise justifies the significance of the term "symmetric".

Question Show that $A \triangle B = B \triangle A$ for any two sets A, B and this is precisely the set of all and only those objects that are elements of exactly one of the sets A, B.

When we work with certain types of objects we usually fix focus on a particular "big" set – the universal set whose objects are primarily our topic of interest. For instance if we are interested about prime numbers then we can focus our attention to the set $\mathbb{N}$, etc.. But note that for someone who is interested about perfect cubes $\mathbb{N}$ cannot serve as a universal set, since there are perfect cubes which are negative.
With respect to the *fixed* universal set $\mathcal{U}$ we define complement of a set as follows :

Definition 1.10. *The* ***complement*** *of a set A is defined to be $\mathcal{U} \setminus A$. And we denote it by writing A^c.*

Definition 1.11. ***Empty set***[6] *is a set which contains no elements.* [7] *We denote an empty set by the Greek alphabet ϕ.*

Finally we define power set of a set as follows :

Definition 1.12. *If X is a set then it's* ***power set*** *$\mathcal{P}(X)$ is defined to be the set of all subsets of X.*

Exercise

1. Show that the operations $\cup$ and $\cap$ are associative.
2. (De Morgan's Laws) Show that for any two sets A, B we must have
   i. $(A \cup B)^c = A^c \cap B^c$
   ii. $(A \cap B)^c = A^c \cup B^c$
3. Show that for any three sets A, B, C the following are true
   i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2 Relations

Definition 2.1. *Given any two sets A, B then the* ***Cartesian product*** *of A with B is $A \times B \overset{def}{=} \{(a, b) : a \in A, b \in B\}$.* [8]

Definition 2.2. *For any two non-empty sets A, B a* ***relation*** *from A to B is a non-empty subset of $A \times B$.*

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[6]sometimes an empty set is also called a null set, though in many branches of mathematics (especially mathematical analysis) the term is used in a different context.

[7]It an be rigorously proved that the empty set is unique (see : ex falso quod libet).

[8]in an ordered pair (or more generally an ordered set) the order of the elements are important. So, for distinct a, b the pairs (a, b) and (b, a) are distinct.

Definition 2.3. *A relation on a non-empty set A is a relation from A to A.*

Definition 2.4. *We say a relation R on set A is*

- ***reflexive*** *if $(a, a) \in R$ for all $a \in A$;*
- ***symmetric*** *if $(a, b) \in R$ implies $(b, a) \in R$;*
- ***asymmetric*** *if it is not symmetric.*
- ***anti-symmetric*** *if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.*
- ***transitive*** *if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$;*

Definition 2.5. *A relation R on a set A that is reflexive, symmetric, and transitive is called an* ***equivalence relation****.*

Definition 2.6. *If R is a relation on a non-empty set A and let $a \in A$ then the set of elements b such that $(a, b) \in R$ is called the* ***equivalence class*** *of a, denoted by $[a]_R$.*

Definition 2.7. *A* ***partition*** *of a set A is a collection of disjoint, non-empty subsets of A whose union is A.*

Theorem 2.8. *The equivalence classes of an equivalence relation R on a set A form a partition of A.*

Proof. Let A be a non-empty set and let R be an equivalence relation on A. For any two $a, b \in A$ we claim that either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \phi$.
To show this, note that if $a, b \in A$ are such that $[a]_R \cap [b]_R \neq \phi$, then there exists some x such that $x \in [a]_R$ and also $c \in [b]_R$. Thus, $(a, x) \in R$, and $(b, x) \in R$. Now, $(b, x) \in R \implies (x, b) \in R$ (since R being an equivalence relation is symmetric). Hence by the transitivity of R we derive $(a, b) \in R$. Therefore, $a \in [b]_R$ and equivalently $b \in [a]_R$.
Further note that by reflexive property of R we get $(a, a) \in R$ for each $a \in A$, and hence for each $a \in A$, we can find some $x \in A$ such that $a \in [x]_R$.
These show that the collection of all distinct equivalence classes of an equivalence relation of R on a set A form a partition of A.

□

Further it's easy to see that the converse of the above theorem is also true. We record it as the following theorem.

Theorem 2.9. *If A is a non empty set and $\mathfrak{P}$ is a partition of A, then the relation R defined on A by : $(x, y) \in R$ if and only if there is some $W \in \mathfrak{P}$ such that $(x, y) \in W$ is an equivalence relation on A.*

Proof. By the way the relation R is defined it is evident that R is reflexive and symmetric. To show that such a relation R must be transitive, note that $(a, b) \in R \implies \exists \ W_1 \in \mathfrak{P}$ such that $\{a, b\} \subseteq W_1$. And $(b, c) \in R \implies \exists \ W_2 \in \mathfrak{P}$ such that $\{b, c\} \subseteq W_2$. Again, since distinct elements of a partition of a set must be disjoint and since $b \in W_1, b \in W_2$, so we must have $W_1 = W_2$. Hence, $a \in W_2$, and thus $(a, c) \in R$. □

3 Functions

Definition 3.1. *A relation R is called a **function** if $(x, y) \in R, \quad (x, z) \in R \implies y = z$.*

Remark 3.2. *Let A, B be two nonempty sets. A **function** f from A to B is a set of ordered pairs such that all the elements of A occur as the first co-ordinates of some element of f, the set of all the second co-ordinate of the elements of f are elements of B and no two distinct elements of f has the same first element.*
We write $f : A \to B$ is a function to mean that f is a function from A to B.

Definition 3.3. *Let $f : A \to B$ be a function. The set A is called the **domain**, and the set of all the second co-ordinates of the elements of f is called the **range** of f.*

A couple of definitions which will be useful :

Definition 3.4. *A function $f : A \to B$ is **injective** if $f(x) = f(y) \iff x = y$. (Sometimes also called* one-one.*)*

Definition 3.5. *A function : $A \to B$ is **surjective** if for all $b \in B$, there is some $x \in A$ such that $f(x) = b$. (Sometimes also called* onto.*)*

Definition 3.6. *A function is **bijective** if it is both injective and surjective.*