Quantum Information Processing and Quantum Computation	02/09/2015

Lecture 5: Superdense Coding

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1 A special unitary operator

Let, $B = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix}$.

Check B is unitary or not. What is $B|\phi^+\rangle$, $B|\phi^-\rangle$, $B|\psi^+\rangle$, $B|\psi^-\rangle$

$$B|\phi^+\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{vmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (|00\rangle)$$

$$B|\phi^-\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{vmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (|10\rangle)$$

$$B|\psi^+\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 1 \\ 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{vmatrix}$$

$$= \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| = |01\rangle$$

$$B|\psi^-\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ -1 \\ 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{vmatrix} = |11\rangle$$

So B is Unitary operator.

2 Superdense coding

Alice and Bob are far away from each other. Alice has two bit a and b. She would like to communicate these two bits to Bob by sending him just a single bit information. In classical world it cannot be possible. Is there any solution in quantum world. Using superdense coding Alice can send two bit of information using single qubit. Alice and Bob must initially share the Bell state, $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Alice is in possession of the first qubit and Bob the second qubit. Alice performs transformation using one of four Pauli matrices, depending on the 2 classical bits she wishes to communicate to Bob. The four pauli matrices are given below.

$$I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}.$$

$$\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

$$\sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

$$\sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}.$$

If Alice wishes to send the bits 00 to Bob, she does nothing to her qubit (or equivalently, applies the identity matrix I. If she wishes to send 01, she applies the σ_x matrix to her qubit. If she wishes to send 10, she applies the σ_z matrix to her qubit and if she wishes to send 11, she applies $\sigma_z\sigma_x$.

To send 00 perform below operation

$$(I \otimes I)|\phi^+\rangle = (I \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{vmatrix}$$

$= \frac{1}{\sqrt{2}} \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|$

$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$

To send 01 perform the following operation

$(\sigma_x \otimes I)|\phi^+\rangle = (\sigma_x \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 \\ 1 \\ 1 \\ 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \left| \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right|$

$= \frac{1}{\sqrt{2}} \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|$

$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$

To send 10 perform the following operation

$(\sigma_z \otimes I)|\phi^+\rangle = (\sigma_z \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$
$= \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ 0 \\ -1 \end{vmatrix}$

$= \frac{1}{\sqrt{2}} \left| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right|$

$= \frac{1}{\sqrt{2}} \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|$

$= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$

To send 11 perform the following operation

$(\sigma_z.\sigma_x \otimes I)|\phi^+\rangle = (\sigma_z.\sigma_x \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \otimes$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 \\ 1 \\ -1 \\ 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \left| \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{2}} \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

You should verify the above states. After applying the appropriate gate(appropriate transformation), Alice sends her qubit to Bob. Then Bob get one of the four Bell states, depending on the classical bits Alice wished to send to him. Bob can now simply perform a measurement of the joint qubit state with respect to the computational basis. first performing a change of basis to the Bell basis, and then performing a measurement in the computational basis. The outcome of the Bell measurement reveals to Bob which Bell state he possesses, and so allows him to determine with certainty the two classical bits Alice wanted to communicate to him.