Quantum Information Processing and Quantum Computation 12/08/2015

Lecture 1: Introduction to Quantum Mechanics and Quantum Information

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1 Introduction

Nature at the sub-atomic scale behaves totally differently from anything that our experience with the physical world prepares us for. Quantum mechanics is the name of the branch of physics that governs the world of elementary particles such as electrons and photons, and it is paradoxical, unintuitive, and radically strange. Below is a sampling of a few such odd features.

- Complete knowledge of a systems state is forbidden - A measurement reveals only a small amount of information about the quantum state of the system.
- The act of measuring a particle fundamentally disturbs its state.
- Quantum entities do not have trajectories. All that we can say for an elementary particle is that it started at A and was measured later at B. We cannot say anything about the trajectory or path it took from A to B.
- Quantum Mechanics is inherently probabilistic. If we prepare two elementary particles in identical states and measure them, the results may be different for each particle.
- Quantum entities behave in some ways like particles and in others like waves. But they really behave in their own unique way, neither particles nor waves.

These features are truly strange, and difficult to accept. As we all know the largest earth-shaking departures from our classical sensibilities are particle-wave duality, and the replacement of absolute determinism with probabilities. The historical development of all of these ideas started through some nice experiments we can look at, to illustrate the concepts. So, we start by describing the experiments that highlights many differences between quantum mechanics and our classical intuition. The intuition gained in the process will help us as we define qubits, and more generally in the study of quantum computing.

2 The Stern-Gerlach Experiment

The Stern-Gerlach experiment is a conceptually simple experiment that demonstrates many basic principles of quantum mechanics. Studying this example has two primary benefits: (1) It demonstrates how quantum mechanics works in principle by illustrating the postulates of quantum mechanics, and (2) It demonstrates how quantum mechanics works in practice through the use of Dirac notation and matrix mechanics to solve problems. By using an extremely simple example, we can focus on the principles and the new mathematics, rather than having the complexity of the physics obscure these new aspects.

In 1922 *Otto Stern* and *Walter Gerlach* performed a seminal experiment in the history of quantum mechanics. In its simplest form, the experiment consists of an oven that produces a beam of neutral atoms, a region of inhomogeneous magnetic field, and a detector for the atoms, as depicted in Figure 1 *Stern* and *Gerlach* used a beam of silver atoms and found that the beam was split into two in its passage through the magnetic field. One beam was deflected upwards and one downwards in relation to the direction of the magnetic field gradient.

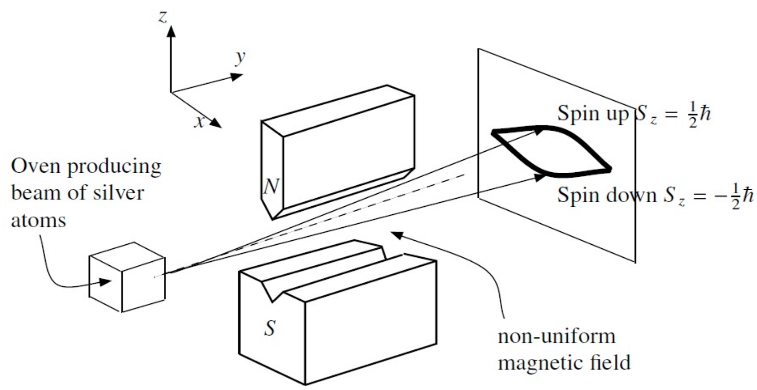


Figure 1: *Stern Gerlach* experiment to measure spin projection of neutral particles along z-axis

To understand why this result is so at odds with our classical expectations, we must first analyze the experiment classically. The results of the experiment suggest an interaction between a neutral particle and a magnetic field. We expect such an interaction if the particle possesses a magnetic moment $\overrightarrow{m}$. A force will be exerted on the particle, and this force is given by

$$\overrightarrow{F} = \overrightarrow{\nabla}(\overrightarrow{m} \cdot \overrightarrow{B})$$

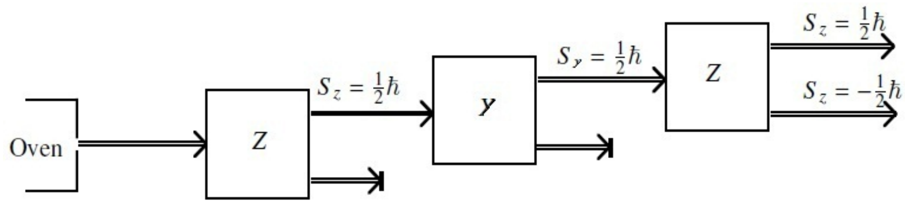
In the *Stern-Gerlach* experiment, the magnetic field gradient is primarily in the z-direction, and the resulting $z$component of the force is given by

$$F_z = \frac{\partial}{\partial z}(m_z \cdot B_z) = m_z \frac{\partial B_z}{\partial z}$$

Let $|\overrightarrow{m}| = M$. Now, if the electron behaves like a classical spinning object, then we should expect the z component of its magnetic moment m_z is expected to take continuous values in $[-M, M]$. However, only finite number of points are registered on screen equally spaced. Sub-atomic particles possesses what we call intrinsic spin angular momentum $\overrightarrow{S}$ and we have $\overrightarrow{m} = \frac{q}{2m}\overrightarrow{S}$. For silver atoms or electrons, $S_z = +\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$, where $\hbar = \frac{h}{2}\pi$, $h =$ Planck's constant $= 6.626 \times 10^{-34}$ Joules-sec. Thus, only two spots are seen on screen.

This result of the *Stern-Gerlach* experiment is evidence of the quantization of the electron's spin angular momentum projection along an axis. This quantization is at odds with our classical expectations for this measurement. In this example, we have chosen the z-axis along which to measure the spin projection, but we could have chosen any other axis and would have obtained the same results.

Let us now consider a series of spin measurements using a sequence of *Stern-Gerlach* devices, as illustrated in following diagram.



In this experiment, atoms are separated in the first device according to their z component of spin. Those for which $S_z = \frac{1}{2}\hbar$ are then passed through a second device in which atoms are separated according to their y component of spin. Those for which $S_y = \frac{1}{2}\hbar$ are passed through a third device which separates the atoms according to their z component, once again. The naive expectation is that, since these atoms have already been preselected to have $S_z = \frac{1}{2}\hbar$, then they will all emerge from the final device in the $S_z = \frac{1}{2}\hbar$ beam. It turns out that this is not what is observed. The atoms emerge randomly in either beam, but with equal probability. The interpretation that immediately comes to mind is that the intervening measurement of the y component of spin has in some way scrambled the z component of spin, but according to classical physics, it should be possible either to arrange the experiment such that any such scrambling be made negligibly small, or else be able to correct for the apparent scrambling in some fashion. It turns out that the quantum effects prevent this from happening this scrambling, and the consequent introduction of randomness into the outcome of the experiment cannot be avoided, except at the cast of not being able to measure the y component of spin at all! Thus we see again an example of intrinsic randomness in the behavior of macroscopic systems.

In the following section, an argument is presented which shows how it is that quantum effects prevent the simultaneous exact measurement of both the y and the z components of spin, i.e., that it is uncontrollable quantum effects that give rise to the scrambling of the z component of spin during the measurement of the y component.

2.1 Incompatible Measurements of Spin Components

The obvious question to ask is whether or not the experiment can be refined in some way to avoid this scrambling. From the perspective of classical physics, the answer is definitely yes, at least in principle. The problem is that the atoms, as they pass through the second *Stern-Gerlach* device, will experience precession about the x axis which will have the effect of changing the z component of the spin. But by suitable fiddling with the beam, the magnetic field strengths and so on it should be possible in principle, at least from the point of view of classical physics, to minimize this effect, or at least determine exactly how much precession occurs, and take account of it. But in practice, it turns out that all these attempts fail. If the experiment is refined in such a manner that the precession is made negligible, (e.g. by using faster atoms, or a weaker magnetic field), the result is that the two emerging beams overlap so much that it is impossible to tell which beam an atom belongs to, i.e. we retain exact information on the z component of spin, but learn nothing about the y component! In general, it appears that it is not possible to measure both S_z and S_x (or, indeed any pair of components of the particle spin), precisely. This kind of behavior is reminiscent of what is found to happen when we attempt to measure both the position and the momentum of a particle. According to the uncertainty principle, the more precisely we determine the position of a particle, the less we know about the momentum of the particle. The difference here is that the quantities being measured are discrete they have only two possible values, whereas the position and momentum of a free particle (even in quantum mechanics) can assume a continuous range of values.

3 Young's Double Slit Experiment

Thomas Young's double slit experiment demonstrates the strange behavior of nature at the atomic level. It is the experiment that we have studied in the high school physics, where it is used to illustrate the wave nature of light. The nature of light was a source of great confusion for the physicists over the centuries. Newton believed that light was a rain of particles, which he called corpuscles. Later there was a lot of evidence that light actually travels as waves. But in the beginning of the twentieth century Einstein discovered the photo-electric effect, which showed that light is transmitted as in discrete packets, which we now call photons. There was a similar kind of confusion with electrons, which were first believed to be particles, but then there was evidence that they behaved like waves in phenomena such as electron diffraction. And so there was a great confusion about the nature of atomic particles; were they wave like or particle like? This confusion did not get resolved until the mid twenties when the laws of quantum mechanics was discovered, which actually showed that atomic particles are neither waves nor particles; they behave in their own strange quantum mechanical way. And it is this strange, counter-intuitive quantum mechanical behavior of atomic particles that we will try to describe through the double slit experiment, which illustrates many of the features of quantum mechanics.

3.1 The Double Slit Experiment with Electrons

In this experiment we have a source of electrons. The intensity of the source is turned down so low, that it emits the electrons as discrete particles once every so often, say once every second or once every fifteen seconds. The apparatus also has a long way from the source a screen with two slits in it and a long way from the screen there is a back-screen along which there is a detector at some point which we measure with the variable x. We are interested in the probability that the electron is detected at the point x on the back-screen.

The experiment is first performed where we block one of the slits. So, if we block slit 2, then the probability that we see the electron at point x varies according to the curve $P_1(x)$, which has the behavior because when the electron goes through slit 1, it gets scattered by the slit and goes in some direction through the slit, but its most likely to go through along a straight line path producing a peak at some neighborhood of x and then the probability drops of in the either side of the neighborhood as we move farther out. There is a similar curve when we block slit 1 keeping slit 2 open and it is denoted by $P_2(x)$. Now if we repeat the experiment with both slits open and graph the probability that the electron went through either slit landed at the position x, we should see the sum of the graph we made for slit 1and a the graph for slit 2 (Figure 2).

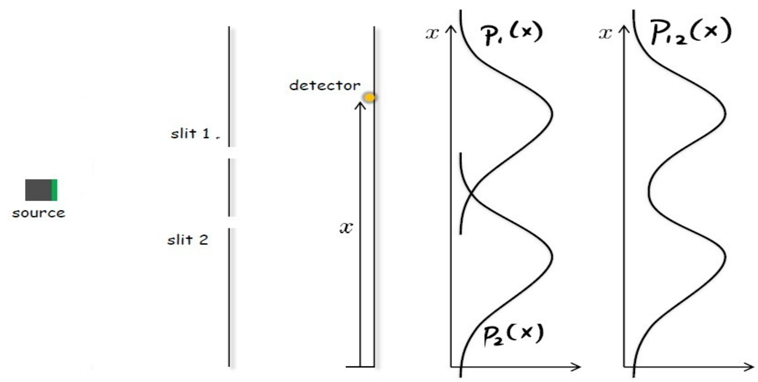


Figure 2: Double slit experiment with electrons: expected observation

But what we actually observe is a interference pattern (Figure 3). The mystery is, how could it possibly be that when both slits are open we get the strange interference pattern, where there are several points at which the probability dropped down very close to 0 as from a substantial probability value at those points when either of the slits are open. This in a nutshell encapsulates the mystery of the behavior of atomic particles.

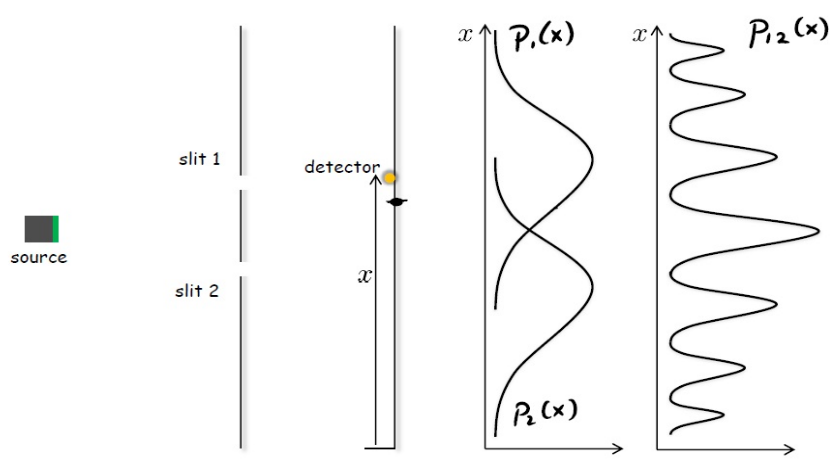


Figure 3: Double slit experiment with electrons: found observation

3.2 The Double Slit Experiment with Bullets

Let us now try to think what would happen to a stream of bullets going through this double slit experiment. The source, which we think of as a machine gun, is unsteady and sprays the bullets in the general direction of the two slits. Some bullets pass through one slit, some pass through the other slit, and others don't make it through the slits. The bullets that do go through the slits then land on the observing screen behind them. Now if we closed slit 2 then the bullets can only go through slit 1 and land in a small spread behind slit 1. If we graphed the number of times a bullet that went through slit 1 landed at the position x on the observation screen, we would see a normal distribution centered directly behind slit 1. If we now close slit 1 and open slit 2, we would see a normal distribution centered directly behind slit 2.

If we repeat the experiment with both slits open and graph the number of times a bullet that went through either slit landed at the position x, we should see the sum of the graph we made for slit 1 and a the graph for slit 2. Let $P_1(x)$ denote the probability that the bullet lands at point x when only slit 1 is open, and similarly for $P_2(x)$. And let $P_{12}(x)$ denote the probability that the bullet lands at point x when both slits are open. Then and we find the graph of Figure 4.

$$P_{12}(x) = P_1(x) + P_2(x) \tag{1}$$

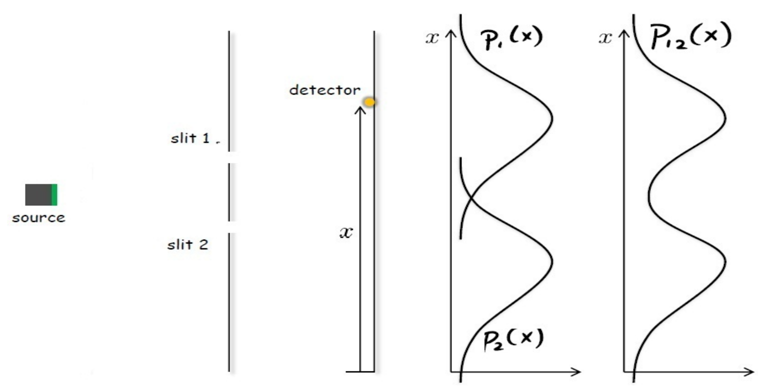


Figure 4: Double slit experiment with bullets: found observation

3.3 The Double Slit Experiment with Water Waves

Next, we consider the situation for waves, for example water waves. In this case we can think of the source as some object that vibrates at a constant rate. A water wave doesnt go through either slit 1 or slit 2, it goes through both. The crest of water wave as it approaches hits the slits, the wave is blocked at all places but the two slits, and waves on the other side are generated at each slit. In the experiment as waves are generated by the source, the energy of the wave at the backstop is detected. We denote the energy function as I and $I(x)$ so is the intensity of the energy function detected at point x. When we plot the energies out, when only slit 1 is open we get $I_1(x)$ and similarly $I_2(x)$ when only slit 2 is open. When both the slits are open, the new waves generated at each slit run into each other, and interference occurs generating the function $I_{12}(x)$. This is exactly the same interference pattern that we have observed in the case of electrons.

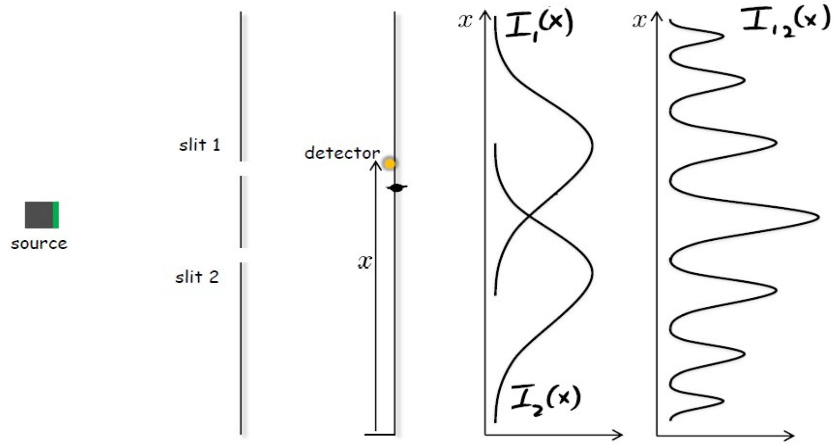


Figure 5: Double slit experiment with water waves: found observation

This differs from the case of bullets. But here we have a very good explanation as why

$$I_{12}(x) \neq I_1(x) + I_2(x) \tag{2}$$

The reason is as follows. The energy of the wave is proportional to $h^2(x)$, where $h(x)$ is the height of the wave at x. Hence

$$I(x) = h^2(x) \tag{3}$$

So, when both slits are open, the height of the wave $h_{12}(x)$ at x is the sum of the waves due to both the slits, which gives

$$h_{12}(x) = h_1(x) + h_2(x) \tag{4}$$

So, the heights add up, but the energies don't, because

$$I_{12}(x) = h_{12}^2(x) = (h_1(x) + h_2(x))^2 \neq h_1^2(x) + h_2^2(x) = I_1(x) + I_2(x) \tag{5}$$

Before we can say what light does with the double slit experiment, we need one more crucial piece of information. What happens when we turn down the intensity in both of these examples?

In the case of bullets, turning down the intensity means turning down the rate at which the bullets are fired. When we turn down the intensity, each time a bullet hits the screen it transfers the same amount of energy, but the frequency at which bullets hit the screen becomes less. With water waves, turning down the intensity means making the wave amplitudes smaller. Each time a wave hits the screen it transfers less energy, but the frequency of the waves hitting the screen is unchanged.

3.4 The Double Slit Experiment with Light

No we see what happens if we perform the same experiment with light.

3.4.1 With Normal Light Intensity

As *Young* observed in 1802, light makes an interference pattern on the screen when the experiment is performed with normal light intensity. From this observation he concluded that the nature of light is wavelike, and reasonably so! However, *Young* was unable at the time to turn down the intensity of light enough to see the problem with the wave explanation. If we consider that the back screen is made of thousands of tiny little photo-detectors that can detect the energy they absorb. For high intensities the photo-detectors individually are picking up a lot of energy, and when we plot the intensity against the position x along the screen we see the same interference pattern described earlier.

3.4.2 With Extremely Reduced Light Intensity for Short Time

Now, turn the intensity of the light very very very low. At first, the intensity scales down lower and lower everywhere, just like with a wave. But as soon as we get low enough, the energy that the photo-detectors report reaches a minimum energy, and all of the detectors are reporting the same energy, say E_0, just at different rates. This energy corresponds to the energy carried by an individual photon, and at this stage we see what is called the quantization of light. Photo-detectors that are in the bright spots of the interference pattern report the energy E_0 very frequently, while darker areas report the energy E_0 at lower rates. Totally dark points still report nothing. This behavior is the behavior of bullets, not waves! We now see that photons behave unlike either bullets or waves, but like something entirely different.

3.4.3 With Extremely Reduced Light Intensity for Sufficiently Long Time

We now turn down the intensity very low for sufficiently long time that only one photo-detector reports something each second. In other words, the source only sends one photon at a time. Each time a detector receives a photon, we record where on the array it landed and plot it on a graph. The distribution we draw will reflect the probability that a single photon will land at a particular point.

Logically we think that the photon will either go through one slit or the other. Then, like the bullets, the probability that the photon lands at a point should be x is $P_{12}(x) = P_1(x)+P_2(x)$ and the distribution we expect to see is the two peaked distribution of the bullets. But this not what we see at all. What we actually see is the same interference pattern from before. But how can this be? For there to be an interference pattern, light coming from one slit must interfere with light from the other slit; but there is only one photon going through at a time! The modern explanation is that the photon actually goes through both slits at the same time, and interferes with itself. The mathematics is analogous to that in the case of water waves. We say that the probability $P(x)$ that a photon is detected at x is proportional to $I(x)$, the light intensity at the point x. Again, $I(x)$ is proportional to

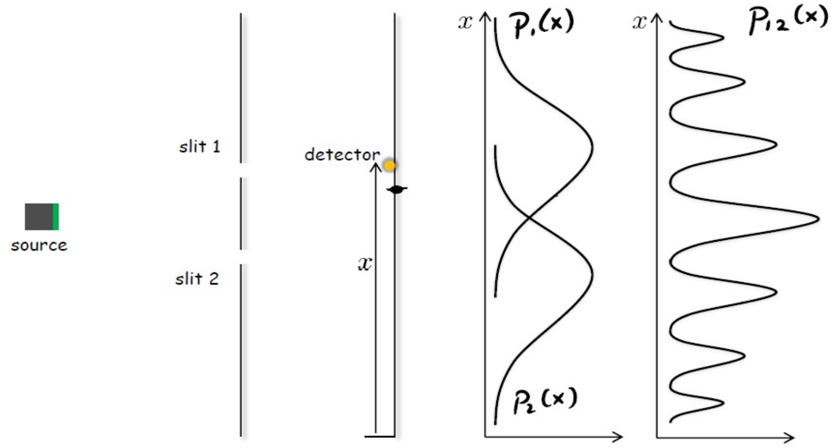


Figure 6: Double slit experiment with light: found observation

the square of some quantity $E(x)$, which we call the amplitude at the point x. Now, the amplitudes for different alternatives add up. So

$$E_{12}(x) = E_1(x) + E_2(x) \tag{6}$$

But

$$\begin{aligned} P_{12}(x) = I(x) = |E_{12}(x)|^2 &= |E_1(x) + E_2(x)|^2 \\ &\neq |E_1(x)|^2 + |E_2(x)|^2 \\ &= I_1(x) + I_2(x) \\ &= P_1(x) + P_2(x) \end{aligned}$$

3.4.4 With Extremely Reduced Light Intensity for Sufficiently Long Time, But Placing a Photon Counter Behind one of The Two Slits

Logically, we can ask which slit the photon went through, and try to measure it. Thus, we might construct a double slit experiment where we put a photon counter at any of the two slits, so that each time a photon comes through the experiment we see which slit it went through and where it hits on the screen. But when such an experiment is performed, the interference pattern gets completely washed out! The very fact that we know which slit the photon goes through makes the interference pattern go away. This is the first example we see of how measuring a quantum system alters the system.

Here the photon looks both like a particle, a discrete package, and a wave that can have interference. It seems that the photon acts like both a wave and a particle, but at the same time it doesnt exactly behave like either. This is what is commonly known as the wave-particle duality, usually thought of as a paradox. The resolution is that the quantum mechanical behavior of matter is unique, something entirely new.

What may be more mind blowing still is, we got the exact same results when we performed the experiment with electrons. Although it is common to imagine electrons as tiny little charged spheres, they are actually quantum entities, neither wave nor particle but understood by their wave function. The truth is that there is no paradox, just an absence of intuition for quantum entities.

4 Postulates of Quantum Computation

Even if we know every minute detail, we can't predict the particle in the experiment discussed. We need an axiomatic approach to match up with the experiments. Quantum mechanics is based upon a set of postulates that dictates how to treat a quantum mechanical system mathematically and how to interpret the mathematics to learn about the physical system in question. These postulates cannot be proven, but they have been successfully tested by many experiments, and so we accept them as an accurate way to describe quantum mechanical systems. The fundamental principles of quantum mechanics are as follows.

a) The state $|\psi\rangle$ of a single particle is a normalized vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$.

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$$

The basis $\{|0\rangle, |1\rangle\}$ is called the standard or the computational basis of the system.

b) A measurement M is denoted by a basis $\{|m_1\rangle, |m_2\rangle\}$ and the outcome of this measurement on the state $|\psi\rangle$ yields two possible cases: The particle in the state $|m_1\rangle$ with probability $\langle m_1|\psi\rangle^2$, or, $|m_2\rangle$ with probability $\langle m_2|\psi\rangle^2$, where $\langle\alpha|\beta\rangle$ is the inner product of two complex vectors defined by

$$\langle\alpha|\beta\rangle = \sum_{i=1}^{n} \alpha_i^* \beta_i$$

where

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

are n dimensional complex vectors and α_i^* is the complex conjugate of α_i.

c) Every meaningful transformation or computation on $|\psi\rangle$ can be represented by a 2×2 unitary matrix U such that

$$|\psi'\rangle = U \cdot |\psi\rangle$$

U is unitary and so $U^\dagger U = UU^\dagger = I$ where $U^\dagger$ is the conjugate transpose of U. Any unitary operator preserves the angles of the input vector.