Lecture 2: Quantum States, Operators, Measurements

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1 Hilbert Space

A state of a particle or a system is given by a vector in a *Hilbert space*. Before we jump into an informal definition of a *Hilbert space*, we discuss some extensions (additional features) of the generic vector space. The extensions are:

- **inner product space:** introduction of the concept of angles between vectors.
- **normed vector space:** introduction of the length of a vector.
- **metric space:** introduction of distance between vectors.

It is interesting to note that one can convert from the inner product space to the normed product space by defining the norm to be the inner product of a vector with itself.

Informally, a *Hilbert space* is a special kind of vector space that, in addition to all the usual rules of the vector spaces, is also endowed with the inner product.
More formally, from

Definition 1.1 *A Hilbert space H is a vector space endowed with an inner product and associated norm and metric, such that every Cauchy sequence in H has a limit in H.*

In the special case where *Hilbert space* has a finite dimension, it is isomorphic to a complex vector space.

Theorem 1.2 (Cauchy-Schwarz inequality)

$$|\langle u|v\rangle|^2 \leq ||u||^2 \ ||v||^2$$

Proof. Let

$$|z\rangle = |u\rangle - \frac{\langle v|u\rangle}{\langle v|v\rangle}|v\rangle$$

We take the inner product with $|v\rangle$, we get

$$\langle v|z\rangle = \langle v|u\rangle - \frac{\langle v|u\rangle}{\langle v|v\rangle}\langle v|v\rangle = 0$$

Hence, $|z\rangle$ is perpendicular to $|v\rangle$.
Now,

$$\langle z|z\rangle \geq 0$$

Hence, from the orthogonality of $|v\rangle$ and $|v\rangle$, we get

$$\langle z|u\rangle \geq 0$$

$$\langle u|u\rangle - \frac{\langle v|u\rangle}{\langle v|v\rangle}\langle v|u\rangle \geq 0$$

$$||u||^2 - \frac{|\langle v|u\rangle|^2}{||v||^2} \geq 0$$

$$||u||^2 \ ||v||^2 \geq |\langle v|u\rangle|^2$$

Hence proved. □

Before we proceed we would like to clarify some notations. $|\psi_1\rangle$ and $|\psi_2\rangle$ are both column vectors. $\langle\psi_1|$ is a row vector that is the conjugate transpose of $|\psi_1\rangle$. While $\langle\psi_1|\psi_2\rangle$ is a row vector multiplied by a column vector resulting in a value, $|\psi_1\rangle\langle\psi_2|$ is a column vector multiplied by a row vector and hence gives a matrix.

Theorem 1.3 (Completeness) *For any state $|\psi\rangle$,*

$$|\psi\rangle\langle\psi| = I$$

Proof.

$$\begin{aligned} \forall|\psi\rangle, \quad |\psi\rangle\langle\psi|\cdot|\psi\rangle &= |\psi\rangle\langle\psi|\psi\rangle && \text{(associativity)} \\ &= |\psi\rangle && \text{(since norm is 1)} \\ &= I|\psi\rangle \\ \therefore |\psi\rangle\langle\psi| = I \end{aligned}$$

□

2 Operators

For quantum, these are restricted to Hermitian (self-adjoint) matrices.

$$A\ |\psi_1\rangle = |\psi_2\rangle$$

Here, A is the operator. A matrix is *self-adjoint* when

$$A = A^\dagger$$

where $A^\dagger$ is the conjugate transpose of A.

Theorem 2.1 *Eigenvalues of hermitian matrices are real.*

Proof. Let A be a hermitian matrix

$$A\ |\psi\rangle = \lambda|\psi\rangle$$

where λ and $|\psi\rangle$ are the eigenvalue and eigenvector respectively.
Now,

$$\begin{aligned} \langle\psi|A\psi\rangle &= \langle\psi|\lambda\psi\rangle \\ &= \lambda\langle\psi|\psi\rangle \end{aligned}$$

We know that $\langle\psi|\psi\rangle$ is real. Now, if we can show $\langle\psi|A\psi\rangle$ to be real, it would ensure λ is real. To show a value is real, we need to show that the complex conjugate is the same as the original value. Hence,

$$\begin{aligned} \langle\psi|A\psi\rangle^* &= \langle A\psi|\psi\rangle \quad \text{(from definition)} \\ &= \langle\psi|A^\dagger\psi\rangle \quad \text{(from definition)} \\ &= \langle\psi|A\psi\rangle \quad (A^\dagger = A) \end{aligned}$$

$\therefore \lambda$ is real. □

Theorem 2.2 *Eigenvectors of hermitian matrices form an orthonormal basis.*

Proof. Assume for simplicity that all the eigenvalues are distinct.
Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues and $|\psi_1\rangle, \cdots, |\psi_n\rangle$ the corresponding eigenvectors. And let A be the given hermitian matrix.
We note that the eigenvectors represent only the direction, and hence can simply be normalized to get unit length. So, our problem of showing orthonormality reduces to showing orthogonality. Hence, we need to show

$$\langle\psi_i|\psi_j\rangle = 0 \quad \forall i \neq j$$

$$\langle\psi_i|A\psi_j\rangle = \langle\psi_i|\lambda_j\psi_j\rangle = \lambda_j\langle\psi_i|\psi_j\rangle \tag{1}$$

$$\langle\psi_i|A\psi_j\rangle = \langle A^\dagger\psi_i|\psi_j\rangle = \langle A\psi_i|\psi_j\rangle = \lambda_i\langle\psi_i|\psi_j\rangle \tag{2}$$

Subtracting 2 from 1, we get

$$0 = (\lambda_j - \lambda_i)\langle\psi_i|\psi_j\rangle \quad \forall i \neq j$$

and since we assumed the eigenvalues to be distinct, we get

$$\langle\psi_i|\psi_j\rangle = 0$$

□

An immediate corollary due to the orthogonality is the following,

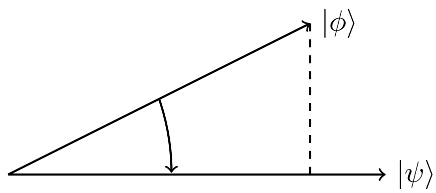
Corollary 2.3 *Any state can be written as a linear combination of the Eigenstates of any hermitian operator in the same Hilbert space (Spectral decomposition).*

Definition 2.4 *An operator A is diagonalizable if $\exists$ a self adjoint operator A' such that*

$$\forall|\psi\rangle \quad A|\psi\rangle = A'|\psi\rangle$$

Theorem 2.5 *Any self-adjoint operator is diagonalizable.*

3 Projectors



Projection of $|\phi\rangle$ into $|\psi\rangle$ is given by

$$\begin{aligned} P_{|\psi\rangle}|\phi\rangle &= \langle\psi|\phi\rangle|\psi\rangle \\ &= |\psi\rangle\langle\psi|\phi\rangle \\ &= (|\psi\rangle\langle\psi|)|\phi\rangle \end{aligned}$$

The first to second equation was possible because $\langle\psi|\phi\rangle$ is a scalar. And hence the projector is defined as

$$P_{|\psi\rangle} \stackrel{\text{def}}{=} |\psi\rangle\langle\psi|$$

We look at projectors onto a subspace spanned by vectors $|\psi_1\rangle, \cdots, |\psi_k\rangle$. The is a k-dimensional subspace.
Let the Hilbert space be of n-dimension, and hence we're projecting n-dimension onto k-dimension. The projector here is defined as,

$$P = \sum_{i=1}^{k} |\psi_i\rangle\langle\psi_i|$$

Corollary 3.1

$$P^2_{|\psi\rangle} = P_{|\psi\rangle}$$

4 Tensor Product

Given two matrices $A_{m\times m} = [a_{ij}]$ and $B_{n\times n} = [b_{ij}]$, the tensor product is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & a_{23}B & \dots & a_{2m}B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & a_{m3}B & \dots & a_{mm}B \end{bmatrix}_{(m\times n)\times(m\times n)}$$

where each element of the above array is a sub-matrix of size $n \times n$.

Postulate 4.1 *If particle* 1 *has a state* $|\psi_1\rangle$ *and particle* 2 *has a state* $|\psi_2\rangle$*, then the joint state of the two particles is given by* $|\psi_1\rangle \otimes |\psi_2\rangle$

Result 4.2 *The tensor product of the basis vectors of two Hilbert spaces* H_1 *and* H_2 *of dimension* m *and* n *respectively forms a basis vector in another Hilbert space of dimension* mn *denoted by* $H_1 \otimes H_2$

Example 4.3 *Consider the computational basis* $\{|0\rangle, |1\rangle\}$ *in* 2*-dimension. Let*

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$

and,

$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

Then the joint state is,

$$\begin{aligned}
|\psi_1\rangle \otimes |\psi_2\rangle \quad &= (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) \\
&= \alpha_1\alpha_2(|0\rangle \otimes |0\rangle) + \alpha_1\beta_2(|0\rangle \otimes |1\rangle) + \\
&\qquad \beta_1\alpha_2(|1\rangle \otimes |0\rangle) + \beta_1\beta_2(|1\rangle \otimes |1\rangle) \\
&= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle
\end{aligned}$$

where $|ij\rangle$ *is the shorthand notation for* $|i\rangle \otimes |j\rangle$*.*

$$\begin{aligned}
|00\rangle \quad &= |0\rangle \otimes |0\rangle \quad = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \quad = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

5 Spectral decomposition of an operator

Previously we've done the spectral decomposition of a vector space, and now we move on to operators. Before we proceed, we will first discuss what a *normal* operator is.

Definition 5.1 *A is normal if and only if*

$$A^\dagger A = AA^\dagger$$

Examples of normal operators are Hermitian operators and unitary operators.

Theorem 5.2 *An operator A has a diagonal representation*

$$A = \sum_{i=1}^{n} \lambda_i |i\rangle\langle i|$$

where λ_i's are the eigenvalues and $|i\rangle$'s are the corresponding eigenstates of A, if and only if A is normal.

6 Generalized version of postulate 2

Any measurement is a collection of operators $\{M_i\}$ acting on the state space such that after the measurement of a state $|\psi\rangle$, outcome i occurs with probability

$$\langle\psi|M_i^\dagger|M_i|\psi\rangle$$

and if the output is i, then the post measurement state becomes:

$$\frac{M_i|\psi\rangle}{\sqrt{\langle\psi|M_i^\dagger|M_i|\psi\rangle}}$$

The denominator of $\sqrt{\langle\psi|M_i^\dagger|M_i|\psi\rangle}$ is added to normalize the new state.
One should also note that the operators $\{M_i\}$ must satisfy the *completeness equation*

$$\sum_i M_i^\dagger M_i = I$$

$$|\psi\rangle \rightarrow \underbrace{\boxed{\{M_1, M_2, \cdots, M_l\}}}_{Measurement} \xrightarrow{Pr(i)=\langle\psi|M_i^\dagger|M_i|\psi\rangle} \underbrace{M_i|\psi\rangle}_{Output\ i}$$

Hence the observed measurement i changes the state to $M_i|\psi\rangle$ and the probability of it being i is $Pr(i)$
<u>Special Case::</u>
Measuring an observable $A = \sum_{i=1} \lambda_i |i\rangle\langle i|$. By axiomatic definition, A is Hermitian, and hence normal. And as seen previously, A thus has a spectral decomposition.
Then,

$$M_i = |i\rangle\langle i| = P_i \text{ (say)}$$

These are nothing but projectors onto the corresponding eigenstates.
<u>Note:</u> Because P_i is Hermitian,

$$M_i^\dagger M_i = P_i^\dagger P_i = P_i P_i = P_i^2 = P_i$$

$\therefore$ the outcome of the measurement is λ_i and the state post measurement is

$$\frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$$

with probability

$$\langle\psi|P_i|\psi\rangle$$

This special case is called the projective measurement.